The Riemannian Penrose Inequality and the Inverse Mean Curvature Flow

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Abstract

In the mathematics of general relativity, the concepts of quasi-local and global mass/energy do not arise as naturally as in Newtonian and Lorentzian physics. While the stress-energy tensor component T_{00} represents the local energy density (of a given observer), the lack of a well-defined gravitational energy density means that the local conservation law for T_{00} familiar from special relativity does not hold in the gravitating case. We discuss this difficulty and show how the global/total mass can be defined assuming asymptotic flatness. The ADM mass provides a definition of the total mass in terms of the three-dimensional Riemannian geometry of a spacelike slice, allowing us to discard the time dependence and Lorentzian signature once we place sufficient constraints on the extrinsic curvature. From a physical perspective, we expect the total mass of a universe to be non-negative, and in the case of a universe containing a black hole (identified by its event horizon) to be at least the usual mass associated a black hole of a given surface area. These two statements (when cast in terms of the intrinsic geometry of a maximal spacelike hypersurface) are respectively known as the Positive Mass Theorem $m \geq 0$ and the Riemannian Penrose Inequality $2m \geq r$, and have been proven in recent decades. We derive the Geroch monotonicity formula for the smooth inverse mean curvature flow and subsequent heuristic proof of the Penrose inequality, and then present the weak formulation introduced by Huisken and Ilmanen that gives a rigorous proof. We also discuss Bray's proof of the Penrose inequality, the generalisation of the Penrose inequality to asymptotically flat solutions of the Einstein-Maxwell equations and why the proof of Bray cannot possibly be generalised in this manner. The application of the weak inverse mean curvature flow to the computation of the Yamabe invariant of 3-manifolds is also discussed, including a sketch of the proof that $\sigma\left(\mathbb{RP}^3\right) = 6\pi^{4/3}$.

Contents

Introduction

This paper is concerned with the *Penrose Inequality*, which is a statement in general relativity that bounds the total mass of a spacetime from below in terms of the horizon area of black holes. When some assumptions are made about the spacetime, the total mass can be expressed in terms of the intrinsic Riemannian geometry of a spacelike hypersurface, and thus the Penrose inequality becomes a statement about three-dimensional Riemannian manifolds. An important case of this inequality can be proven using a weak formulation of a geometric flow known as the *Inverse mean curvature flow* (*IMCF*). We will see the details of this proof and other applications of the flow, both to generalisations of the Penrose inequality and to other topics in Riemannian geometry.

In Section 1, we discuss the difficulties that arise in the definition of mass/energy in general spacetimes. The nature of gravitation in general relativity is radically different to the simple potential/force model of Newtonian gravitation; in particular, there is no notion of local gravitational energy density, so we cannot define a total energy density satisfying the usual continuity equation. However, making some assumptions about the asymptotic geometry of the spacetime gives a well-defined *total* mass, known as the ADM mass. It agrees with the usual notion of mass for the symmetric black hole spacetimes (Schwarzschild, Kerr, Reissner–Nordström) and is expressed in terms of the acceleration of initially stationary observers at infinity, where the spacetime looks like a point mass. Assuming the existence of a maximal spacelike hypersurface, we arrive at a precise statement of the Riemannian Penrose Inequality for 3-manifolds of positive scalar curvature.

Section 2 discusses the initial heuristic proof of the Penrose inequality. It was noticed by Geroch that the quantity known as the *Hawking quasi-local mass* was non-decreasing when surfaces were flowed outwards by the inverse of their mean curvature. Assuming the existence of such a flow starting at the black hole horizon and expanding towards coordinate spheres asymptotically, the Penrose Inequality holds, as pointed out by Jang and Wald. However, the existence of such a flow is not guaranteed, with some quite simple counterexamples easily found.

Section 3 discusses the weak formulation of the flow introduced by Huisken and Ilmanen. By allowing the surfaces to jump outwards at certain times, they avoid the singularities of the smooth flow and have a guaranteed solution with the desired asymptotics; and (by careful choice of the minimisation scheme to control the geometry at the jump times) the monotonicity of the Hawking mass can be preserved. We will present most of the details of the complete proof.

Section 4 has a more broad scope, and discusses generalisations of the Penrose inequality and other applications of the IMCF. One generalisation of the inequality (that allows disconnected horizons) cannot be proven by modifications of the IMCF argument, even though it is known to be true via a different proof due to Bray. The opposite is true for the generalisation to charged black holes - the argument of Huisken and Ilmanen can easily be generalised, while the analogous statement that a modification of Bray's proof (i.e. allowing disconnected horizons) would produce is demonstrably false. The IMCF also has an application in computing the so-called Yamabe Invariant of compact 3-manifolds; so we will see that Bray's proof in no way obsoletes the utility of the IMCF.

The reader should have a basic knowledge of (semi-)Riemannian geometry, special & general relativity and functional analysis. Many of the arguments at the core of the weak formulation rely on geometric measure theory and the existence and regularity of solutions to linear elliptic PDEs, but a detailed knowledge of these topics is not necessary.

1 Mass in General Relativity

1.1 Energy Conservation

Conventions: we use natural units where $c, G = 1$. The signature of the spacetime metric is $(-,+,+,+)$.

In special relativity, the classical notion of mass/energy is easily defined and satisfies useful conservation laws as in pre-relativistic physics. We have the local mass/energy as encapsulated by the Stress-energy tensor T, which is a conserved current ($\partial^\mu T_{\mu\nu} = 0$) as a consequence of Noether's theorem and the invariance of physical Lagrangians under spacetime translations. Fixing an observer (i.e. a Lorentzian coordinate system), a compact region of space $K \subset \mathbb{R}^3$ and a time interval $[t_0, t_1]$, we have (by the divergence theorem)

$$\begin{aligned} 0 &= \int_{K\times[t_0,t_1]} \partial^\mu T_{\mu 0}\, dV\, dt \\ &= \int_{K\times\{t_1\}} T_{00} dV - \int_{K\times\{t_0\}} T_{00} dV + \int_{t_0}^{t_1} \int_{\partial K} T_{\mu 0}\nu^\mu dA\, dt \end{aligned}$$

where ν is the outwards normal to K. The first two terms are the total energy inside K at times t_0 and t_1 and the last term is the integrated outwards energy flux through ∂K, so this is local energy conservation

$$\text{Final Energy} = \text{Initial Energy} - \text{Total energy lost through boundary.}$$

If we assume the stress-energy tensor has compact support on the a spatial slice, then for K containing this support (i.e. K containing all matter in the universe) the flux term vanishes and we find

$$\frac{dE}{dt} = \frac{d}{dt}\int_{K\times\{t\}} T_{00} dV = 0;$$

i.e. the total energy is conserved. When instead the energy flux does not have compact support but dies off sufficiently fast (as r^{-2}) at spatial infinity, we can take the limit $r \to \infty$ with $K = B_r$ to arrive at the same result.

When we move to general relativity, we replace the Minkowski spacetime with an arbitrary Lorentzian 4-manifold (L, g), and the condition on the stress-energy tensor becomes $\nabla^\mu T_{\mu\nu} = 0$ for ∇ the covariant derivative of (L, g). For the purposes of measuring lcoal energy density, an arbitrary observer is now represented by a timelike vector field $\xi = \partial/\partial t$. In our argument for energy conservation, we used the fact that the divergence of the energy flux is

$$\partial^\mu T_{\mu 0} = \partial^\mu \left(T_{\mu\nu}\delta_0^\nu\right) = \left(\partial^\mu T_{\mu\nu}\right)\delta_0^\nu = 0.$$

If we now make the natural replacements $\partial \to \nabla$ and $\delta_0^\nu \to \xi^\nu$, we instead find

$$\nabla^\mu \left(T_{\mu\nu}\xi^\nu\right) = \left(\nabla^\mu T_{\mu\nu}\right)\xi^\nu + T_{\mu\nu}\nabla^\mu\xi^\nu = T_{\mu\nu}\nabla^\mu\xi^\nu;$$

so local conservation of energy only holds in general if the symmetric part of $\nabla\xi^\flat$ is zero; i.e. ξ is a Killing field. (The argument is identical to the Minkowski case, where the 4-dimensional cylindrical region is replaced by the region swept out by a hypersurface under the flow of ξ; i.e. $K \times \{t\}$ is replaced by $\Phi^\xi_{t-t_0} K_0$ for K_0 some initial spacelike hypersurface with boundary.) The physical justification for this failure is that there is now energy stored in the "gravitational field" that is not accounted for in the stress-energy tensor. Since the Newtonian gravitational energy density is proportional to $|\nabla\Phi|^2$ (for Φ the potential) and Φ is proportional to $g_{00} - \eta_{00}$ in the Newtonian limit [2, Section 4.4a], any candidate for a general relativistic gravitational energy density should be expressed in terms of the first covariant derivative of the metric. But $\nabla g = 0$, so by moving to the geometric theory we have lost the concept of local gravitational potential energy, and thus cannot define a meaningful conserved local energy density.

1.2 Total Energy

Despite the issue raised in the previous section, in turns out that in certain cases there is a well-defined *total* mass for entire systems. We begin by looking at the case of Newtonian gravity, where the total mass is easily defined as

$$m = \int_K \rho \, dV$$

for the mass density ρ and a set K containing the support of ρ. We can write this using Poisson's equation $4\pi\rho = \Delta\Phi$ (Φ the gravitational potential) and the divergence theorem as

$$\begin{aligned} m &= \int_{B_r} \rho \, dV \\ &= \frac{1}{4\pi} \int_K \Delta\Phi \, dV \\ &= \frac{1}{4\pi} \int_K \nabla\Phi \cdot \nu \, dA \\ &= -\frac{1}{4\pi} \int_K a \cdot \nu \, dA \end{aligned} \tag{1.1}$$

where a is the acceleration due to gravity and ν is the outwards unit normal to K. In the case where ρ is not compactly supported but decays sufficiently fast at infinity, we can take an exhaustion of $\mathbb{R}^3$ by spheres, giving the limit

$$m = \int_{\mathbb{R}^3} \rho \, dV = \lim_{r\to\infty} -\frac{1}{4\pi} \int_{B_r} a \cdot \nu \, dA.$$

We can now interpret $a \cdot \nu$ as the outwards force done to hold a unit mass in place against the force of gravity. Thus we can translate this definition of mass into general relativity when we have notions of "holding in place" and of a "large sphere". Formally, this means we want a *stationary* spacetime L (one with a global time translation symmetry $\xi = \partial/\partial t$, $\nabla_{(\mu}\xi_{\nu)} = 0$) with a foliation of spacelike slices M_t (such that $M_{t+\delta} = \Phi^{\xi}_{\delta} M_t$) that are *asymptotically flat* (i.e. approach flat space at spacelike infinity).

To generalise equation 1.1 to the case of general relativity, we can take S to be a topological sphere inside one of the spacelike slices enclosing the support of the stress-energy tensor and a to be the acceleration of an observer following the time translation ξ, which is $a = \frac{1}{V}\nabla_\xi \xi$, $V = |\xi|$ [2]. If we assume the spacetime is *static* (i.e. the slices M_t are orthogonal to ξ), then the sphere S has unit normals ν (in the spacelike direction) and ξ/V (timelike), so the canonical volume 4-form is $d\mu = dA \wedge \nu^\flat \wedge \xi^\flat / V$. Thus (remembering that ξ is a Killing field and therefore satisfies

$\nabla_j \xi_k = \nabla_{[j}\xi_{k]})$

$$\begin{aligned} m &= -\frac{1}{4\pi}\int_S \langle a, \nu\rangle \, dA \\ &= -\frac{1}{4\pi}\int_S \nu_k \frac{1}{V}\xi_j \nabla^j \xi^k dA. \\ &= -\frac{1}{4\pi}\int_S \nabla^j \xi^k \nu_{[k}\xi_{j]} V^{-1} dA \\ &= -\frac{1}{8\pi}\int_S \nabla^j \xi^k d\mu_{jklm} dx^l \wedge dx^m \\ &= -\frac{1}{8\pi}\int_S \star d\xi. \end{aligned}$$

This is known as the *Komar Integral.* Since

$$d \star d\xi = \star\Delta\xi = \frac{2}{3}\star\left(\mathrm{Rc}\,(\xi,\cdot)^{\sharp}\right) \tag{1.2}$$

[2] which vanishes in vacuum regions by Einstein's equation, Stoke's theorem tells us the Komar integral is the same for any two homologous spheres enclosing the support of the stress-energy tensor; so in the case when the matter content has finite extent we can use this as the definition of the total mass. Noting that $8\pi m = -\int \star d\xi$ is a coordinate-free expression, we can therefore discard the requirement that the spacetime be static and use this expression for the mass of any stationary spacetime. As in the Newtonian case, we can generalise this by taking a limit $r \to \infty$ and requiring the stress-energy tensor to decay sufficiently fast at spatial infinity; call the limiting value of m the *Komar Mass.* In the case where S is eventually the (full) boundary of a compact region B (e.g. when the universe has Euclidean topology), we can use Stoke's theorem, (1.2) and Einstein's equation to write the Komar mass in terms of the stress-energy tensor:

$$m = -\frac{1}{8\pi}\int_B d\star d\xi = 2\int_B \left(T - \frac{1}{2}\mathrm{tr}\,(T)\,g\right)(\nu,\xi)\,;$$

but for general spacetimes, no such volume integral expression exists. The Komar mass can be generalised further to non-static spacetimes that have an *asymptotically* Killing timelike vector field [2]. However, since we are interested in the mass at a "single moment in time", the dataset we actually want to work with is (M, g, k) where M is a 3-manifold taken as a maximal spacelike slice of the spacetime L, g is the induced Riemannian metric on M and k is the second fundamental form of $M \hookrightarrow L$. The condition of *maximality* means tr $k = 0$. This would allow everything to be done in the framework of Riemannian geometry. It turns out that we can rewrite the Komar mass in terms of the 3-dimensional geometry [3], giving an expression known as the ADM mass:

$$m_{\mathrm{ADM}} = \frac{1}{16\pi}\sum_i \int_{\partial B_r} \left(g_{ij,i} - g_{ii,j}\right)\nu^j dA$$

where the metric components and area form are defined in terms of an asymptotically flat coordinate system - we will make this precise soon.

Arnowitt, Deser and Misner (ADM) took the approach of describing general relativity in terms of a foliation by spacelike 3-manifold slices. Writing the usual Einstein-Hilbert Lagrangian $\mathcal{L} = R\sqrt{g}$ in terms of the foliation variables, they arrived at a Hamiltonian formulation of general relativity. By applying a slight generalisation of Noether's Theorem (necessary because the Lagrangian uses second-order derivatives of the metric) to the time-translation symmetry ξ, we arrive at the exact same expression for the ADM mass. [6]

Definition 1.1. A Riemannian manifold (M, g) is *strongly asymptotically flat* if there is a compact set K and coordinates x^i on $M \backslash K$ such that

$$|g_{ij} - \delta_{ij}| \in O(1/r) \quad |g_{ij,k}| \in O\left(1/r^2\right)$$

as $r = \sqrt{\sum_i x^i x^i} \to \infty$, where $\delta = \operatorname{diag}(1,1,1)$ is the standard Euclidean metric for the coordinates x^i.

Definition 1.2. The *ADM mass* of an asymptotically flat manifold is

$$m_{\text{ADM}} = \frac{1}{16\pi} \lim_{r\to\infty} \sum_i \int_{\partial B_r} (g_{ij,i} - g_{ii,j}) \nu^j dA \tag{1.3}$$

where ν is the outwards unit normal field to the 2-sphere $\partial B_r(0) = \left\{\sum_i x^i x^i = r^2\right\}$, dA is the area form induced by the flat metric on ∂B_r and the tensor components are with respect to the asymptotic coordinates x^i.

Proposition 1.3. *[4] The ADM mass is well-defined in strongly asymptotically flat manifolds; i.e. the limit converges and is independent of the coordinate system chosen - if x^i and y^i are two coordinate systems satisfying the asymptotic conditions, then they both give the same value for m_{ADM}.*

The total mass essentially measures the asymptotic rate at which the gravitational field drops off; so since the analogue of the gravitational potential in general relativity is the metric, the ADM mass measures the asymptotic rate at which the space approaches Euclidean space. We make this precise with the following examples.

Example 1.4. In the Newtonian case, Φ is harmonic in the vacuum region and thus we can expand Φ (assuming it is $O\left(r^{-1}\right)$ to ensure convergence of total mass) as a multipole expansion

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{ml} r^{-l-1} Y_l^m(\theta, \varphi)$$

where Y_l^m are the normalised spherical harmonics on S^2. Thus we have

$$\begin{aligned} 4\pi m &= \lim_{r\to\infty} \int_{S_r} \frac{\partial \Phi}{\partial r} dA \\ &= \lim_{r\to\infty} \sum_l \sum_m (-l-1) C_{ml} r^{-l-2} \int_{S_r} Y_l^m r^2 d\Omega \\ &= \lim_{r\to\infty} \sum_l \sum_m (-l-1) C_{ml} r^{-l} \int_{S_r} Y_l^m d\Omega \\ &= -\sum_{m=-0}^{0} C_{m0} = -C_{00} \end{aligned}$$

i.e. the total mass is just the monopole coefficient.

Example 1.5. In the case where the metric is conformally flat to first order

$$g_{ij} = \left(1 + \alpha r^{-1}\right) \delta_{ij} + O\left(r^{-2}\right)$$

we can easily make the analogous statement for general relativity precise: we have

$$g_{ij,k} = -\frac{\alpha x_k}{r^3}\delta_{ij} + O\left(r^{-3}\right)$$

and therefore

$$\begin{aligned}
\sum_i g_{ij,i} n^j &= -\sum_i \frac{\alpha x_i}{r^3} n_i + O\left(r^{-3}\right) = -\frac{\alpha}{r^2} + O\left(r^{-3}\right) \\
\sum_i g_{ii,j} n^j &= -\sum_i \delta_{ii} \frac{\alpha x_j}{r^3} n^j + O\left(r^{-3}\right) = -\frac{3\alpha}{r^2} + O\left(r^{-3}\right).
\end{aligned}$$

Plugging these into the definition of the ADM mass gives

$$\begin{aligned}
m_{\mathrm{ADM}} &= \frac{1}{16\pi} \lim_{r\to\infty} \left(\int_{\partial B_r} \frac{2\alpha}{r^2} dA + \int_{\partial B_r} O\left(r^{-3}\right) dA \right) \\
&= \frac{1}{16\pi} \lim_{r\to\infty} \left(4\pi r^2 \frac{2\alpha}{r^2} + O\left(r^{-1}\right) \right) \\
&= \frac{\alpha}{2}
\end{aligned}$$

exactly; so up to the factor of 2, the ADM mass is precisely the r^{-1} decay rate of the deviation from the flat metric. In the Schwarzschild case, the metric is (in isotropic coordinates) $(1+m/2\rho)^4 \delta = \left(1 + 2m/\rho + O\left(\rho^{-2}\right)\right)\delta$, so we recover the Schwarzschild mass parameter as the ADM mass.

In what follows, the non-negativity of scalar curvature will be an integral component of some arguments. This proposition justifies the assumption by showing it to be true for the case of maximal spacelike hypersurfaces of physically reasonable spacetimes.

Proposition 1.6. *Let $(M, \widetilde{g}, k)$ be a maximal spacelike hypersurface of a spacetime (L, g). If (L, g) satisfies the weak energy condition then $(M, \widetilde{g})$ has non-negative scalar curvature.*

Proof. The weak energy condition states that $T(X,X) = \mathrm{Rc}(X,X) - \frac{1}{2}Rg(X,X) \geq 0$ for any future-directed timelike vector X. In particular, consider the future-pointing unit normal ν to M; then we have $T(\nu,\nu) = \mathrm{Rc}(\nu,\nu) + \frac{1}{2}R \geq 0$. Now write the Gauss equation for the scalar curvature $\widetilde{R}$ of the submanifold:

$$R_{ijkl} = \tilde{R}_{ijkl} + k_{ik}k_{jl} - k_{jk}k_{il}.$$

Since we are interested in the submanifold scalar curvature $\widetilde{R}$, we want to take 3-dimensional traces; i.e. contractions with the 3-metric $\widetilde{g}_{ij} = g_{ij} + \nu_i \nu_j$:W

$$\begin{aligned}
\mathrm{Rc}_{jl} + R_{\nu j \nu l} &= \widetilde{\mathrm{Rc}}_{jl} + k_i^i k_{jl} - k_k^i k_{il} \\
R + 2\mathrm{Rc}(\nu,\nu) &= \widetilde{R} + H^2 - |k|^2
\end{aligned}$$

where $H = \mathrm{tr}k$ is the mean curvature of $M \hookrightarrow L$ and $|k|^2 = \mathrm{tr}\left(k^2\right)$. We see now that since $H = 0$ (M is maximal), the weak energy condition says exactly that

$$\widetilde{R} \geq |k|^2 \geq 0.$$

□

Note. We used above (and will continue to use throughout this paper) the convention that the mean curvature is the trace of the second fundamental form. Classically this was divided by the hypersurface dimension (to produce the average of the principal curvatures, hence the name mean).

1.3 Positivity of Mass

Physically, we expect that the total energy of an isolated gravitating system should always be positive; but this is not at all evident from the definition of the ADM mass. This was known as the positive mass conjecture. The work of Schoen and Yau in the 1970s led to its proof:

Theorem 1.7. *If (M, g) is asymptotically flat with non-negative scalar curvature, then $m_{\mathrm{ADM}} \geq 0$ with equality if and only if M is isometric to $\mathbb{R}^3$.*

Proof. We give a sketch. Assume $m < 0$. Then by a conformal transformation we can construct a metric $\tilde{g}$ such that $\tilde{R} > 0$ and $\tilde{m} < 0$. Consider the coordinate circles $C_\sigma := \left\{(x_1, x_2, x_3)\,|x_1^2 + x_2^2 = \sigma^2\right\}$. Each C_σ is spanned by an area-minimising surface S_σ. One can show that for sufficiently large h, the surfaces S_σ are contained in the cylinders $\left\{x_1^2 + x_2^2 \leq \sigma^2, |x_3| \leq h\right\}$. Applying the regularity theory for minimal surfaces gives compactness, and thus taking a sequence $\sigma_n \to \infty$ we can extract a subsequence $\sigma_{n_k} \to \infty$ such that $S_{\sigma_{n_k}}$ converges in C^2 on compact sets to a complete area-minimising surface bounded between the planes $x_3 = \pm h$. For a compact variation $f\nu$ of S, we have the second variation formula (with $\mathrm{Rc}\,(\nu, \nu)$ replaced via the Gauss equation)

$$\frac{d^2}{dt^2}|_{t=0} A\left(\Phi_t^{f\nu}\left(S \cap \mathrm{supp} f\right)\right) = \int_{S \cap \mathrm{supp} f} -f\Delta f + f^2\left(\kappa - \frac{1}{2}|k|^2 - \frac{1}{2}R\right) dA \geq 0$$

since S is area-minimising. (Δ is the Laplacian derived from the induced metric on S). Choosing an appropriate f, this yields

$$\int_S \kappa\, dA > 0.$$

An argument using the Gauss-Bonnet theorem (with boundary term) on the discs $B_\rho \cap S$ as $\rho \to \infty$ shows that

$$\int_S \kappa\, dA \leq 0,$$

a contradiction. □

For a full proof including the existence of $\tilde{g}$ and the construction of the variation f, see the original paper [5] or the exposition in [6].

1.4 Horizons and Trapped Surfaces

One of the most radical predictions of general relativity is the formation of *black holes* by gravitational collapse. Black holes are regions of spacetime from which no light rays (and thus no matter or information/causal effects at all) can escape to future lightlike infinity[1]. The *event horizon* is the boundary of a black hole; i.e. the "point of no return". In the case of a spherically symmetric mass of radius r_m in an otherwise vacuum spacetime, the region of spacetime outside the extent of the mass (i.e. $r > r_m$) is described by the usual Schwarzschild metric. When the radius of the mass is less than the Schwarzschild radius $r_S = 2m$, the surface $r = r_S$ bounds a black hole. By the singularity theorem of Hawking and Penrose [7], a physically reasonable spacetime (satisfying some energy conditions and containing no closed timelike curves) containing a black hole will eventually

[1] There is a precise definition of an asymptotically flat spacetime using a compactification that provides points at spatial infinity, past and future timelike infinity and a space of past and future lightlike infinities (for the various directions one can escape in).

develop a *singularity*; i.e. a failure of geodesic completeness that cannot be removed by extending the spacetime; so even if initially we have a smooth spherically symmetric matter distribution, if it is sufficiently dense then it will collapse to a point, giving the Schwarzschild slice after this time.

The definition of the event horizon is fundamentally global and intertwined with the 4-dimensional geometry of the full spacetime; so in general it cannot be defined in terms of the slice data (M, g, k). However, in the stationary case, the intersection of the event horizon with M is precisely the (locally defined) *apparent horizon* [8, 9.3.1], which we will now define.

Definition 1.8. A *trapped surface* is a surface $N \hookrightarrow M \hookrightarrow L$ such that the expansion in the normal future lightlike directions is negative; i.e. $\mathrm{tr}_N \nabla X < 0$ for any lightlike vector field X normal to N.

A *marginally trapped surface* replaces the strict inequality with ≤ 0.

$N = \partial C$ is an *outer marginally trapped surface* if it is the boundary of a compact set C in M and the marginally trapped inequality holds for X such that $g(X, \nu_N) \geq 0$.

In this case, C is called a *trapped region.*

Here the vector fields X are the generators of flows that represent the trajectories of photons moving orthogonally from N; so the trapped surface condition means that both the "inwards" and "outwards" light fronts decrease in area (at least locally). In the time-symmetric case (and in particular the static case), these generators are (scalings of) $\xi \pm \nu$ so we have

$$0 < \mathrm{tr}_N (\nabla \xi) \pm \mathrm{tr}_N (\nabla \nu) = \pm H$$

(where $H = \mathrm{div}_N \nu$ is the mean curvature of N in M) since the fact ξ is a Killing field implies $\nabla \xi$ is antisymmetric. That is, the future outwards lightlike expansion is exactly the mean curvature.

Proposition 1.9. *If the closure of the union of all trapped regions in M is a smooth manifold with boundary, then its boundary has zero expansion in the future lightlike directions. [2]*

Definition 1.10. Under the hypothesis of Proposition 1.9, the *apparent horizon* of M is

$$\mathcal{A} = \partial \left(\overline{\bigcup \{\text{trapped regions in } M\}} \right).$$

In the static case, the union of trapped regions is the black hole, and thus $\mathcal{A}$ is the event horizon. The fact that $\mathcal{A}$ has zero expansion means that (assuming time symmetry) it has zero mean curvature in M; i.e. it is a minimal surface of M. Since we took the union of *all* trapped regions in M, $\mathcal{A}$ is the *outermost* compact minimal surface in M. We will use this as our definition of the horizon.

1.5 The Penrose Inequality

While considering the formation of black holes, Penrose originally conjectured [11] (and gave a physical argument for) an inequality between the (apparent horizon) surface area $A(N)$ of a black hole and the total ADM mass m_{ADM} of the spacetime containing it:

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A(N)}{16\pi}}.$$

Writing $A = 4\pi r^2$, this is $2m \geq r$; so we see that the inequality is saturated in the Schwarzschild case. Since we can always choose the spacetime to be vacuum outside the black hole, this is best interpreted as being a bound on the area of a black hole of given mass. Moving to the Riemannian picture, the black hole horizon becomes an outermost minimal surface in the spacelike slice M, and we arrive at the full statement of the Riemannian Penrose Inequality:

Theorem 1.11. *Let M be an asymptotically flat, connected, complete Riemannian 3-manifold with non-negative scalar curvature. If N_0 is an outermost compact minimal surface of M (i.e. there are no compact minimal surfaces of M enclosing N_0) then*

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{A\left(N_0\right)}{16\pi}} \tag{1.4}$$

where m_{ADM} is the ADM mass of M. Furthermore, if the inequality is saturated, then (M, g) is isometric to the Schwarzschild slice.

Since the region enclosed by N_0 plays no role in either the theorem or its proof, we can discard the interior and assume without loss of generality that $N_0 = \partial M$. While this full version of the theorem has been proven, it cannot be proven by the techniques presented in this paper - see Section 4.1 for a discussion. We will instead prove the weaker version when the horizon N_0 is connected:

Theorem 1.12. *Let M be an asymptotically flat connected complete Riemannian 3-manifold with non-negative scalar curvature. If ∂M is compact, connected and a minimal surface, and there are no compact minimal surfaces in the interior of M then the Penrose inequality holds for $N_0 = \partial M$. Furthermore, if the inequality is saturated, then (M, g) is isometric to the Schwarzschild slice.*

Proof. We will define the Hawking mass $m_H(E) \in \mathbb{R}$ of a surface E and a flow of connected surfaces N_t starting at N_0 such that

1. $t \mapsto m_H(N_t)$ is non-decreasing,
2. $m_H(N_0) = \sqrt{A(N_0)/16\pi}$, and
3. $m_H(N_t) \to m_{\mathrm{ADM}}$ as $t \to \infty$.

We then have $m_{\mathrm{ADM}} \geq m_H(N_t) \geq m_H(N_0) = \sqrt{A(N_0)/16\pi}$ as desired. For the rigidity, see Proposition 3.16. □

We will present the proof of this theorem given by Huisken and Ilmanen [1]. The central ingredient is the monotonicity of the Hawking mass first used by Geroch [9] in a proof of the Positive Mass Theorem (assuming a smooth inverse mean curvature flow starting from small spheres). This was similarly applied to the Penrose inequality by Jang and Wald [10], under the assumption that the flow N_t remained smooth. The contribution of Huisken and Ilmanen was to define a generalised weak flow with guaranteed existence that still satisfies the required conditions.

2 Smooth Inverse Mean Curvature Flow

This section will introduce the Inverse Mean Curvature Flow and detail the Jang-Wald approach to the Riemannian Penrose Inequality using the Hawking mass. The only missing ingredient here is the existence of the smooth flow N_t; and we will see in general that it does not exist, necessitating the reformulation in Section 3. We will continually refer back to the canonical example of the Schwarzschild spacelike slice, with initial condition being the event horizon.

2.1 Definition

We start with an intuitive definition of a geometric flow. The motivation behind its introduction for this problem is Geroch's monotonicity result, which will be proved later.

Definition 2.1. Let I be an interval and U a smooth 2-manifold. A smooth function $x : U \times I \to M$ is a solution of the classical/smooth inverse mean curvature flow (IMCF) if and only if the velocity field $X := \partial x / \partial t = dx\left(\partial_t\right)$ satisfies

$$X = \frac{1}{H}\nu \tag{2.1}$$

where $H\left(p,t\right)$ is the mean curvature of $N_t = x\left(U,t\right)$ at p and $\nu\left(p,t\right)$ is the outwards unit normal to N_t at p.

Example 2.2. The simplest example of the IMCF is for a round sphere in Euclidean 3-space. The mean curvature of a sphere of radius r is $2r^{-1}$, so the sphere flows outwards at a uniform speed $r/2$. Thus the flow consists of concentric spheres $N_t = S_{r(t)}$ satisfying

$$\frac{dr}{dt} = \frac{r}{2}$$

which has solution $r\left(t\right) = r\left(0\right)e^{t/2}$.

2.2 Geometric Evolution

Let us investigate the evolution of the submanifold geometry under the smooth IMCF. We can compute derivatives of various quantities using the flow Φ_t^X of the normal velocity field X, since $N_t = \Phi_t^X N_0$. First, consider the area $A\left(N_t\right)$. We find

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} A\left(N_t\right) &= \left.\frac{d}{dt}\right|_{t=0} \int_{\Phi_t^X N_0} dA \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{N_0} \Phi_t^{X*} dA \\
&= \int_{N_0} \left.\frac{d}{dt}\right|_{t=0} \Phi_t^{X*} dA \\
&= \int_{N_0} L_X dA \\
&= \int_{N_0} H\left|X\right| dA
\end{aligned}$$

where we used the result of Proposition B.1. Since the evolution equation (2.1) is invariant under $t \mapsto t + \Delta t$, we therefore have

$$\begin{aligned} \frac{d}{dt} A\left(N_t\right) &= \int_{N_t} H\left|X\right| dA \\ &= A\left(N_t\right) \end{aligned} \tag{2.2}$$

since $|X| = 1/H$ for IMCF.

Now consider the evolution of the mean curvature. We will write $v = |X|$ for the speed.

$$\begin{aligned} \frac{1}{v}\frac{\partial H}{\partial t} &= \frac{1}{v}\nabla_{v\nu} H \\ &= \nabla_\nu \mathrm{tr} \nabla \nu \\ &= \mathrm{tr} \nabla^2_{\nu,\cdot} \nu \\ &= -\mathrm{Rc}\left(\nu,\nu\right) + \mathrm{tr}\nabla^2_{\cdot,\nu}\nu \\ &= -\mathrm{Rc}\left(\nu,\nu\right) + dx^i \nabla_i \nabla_\nu \nu - dx^i \nabla_{\nabla_i \nu} \nu \\ &= -\mathrm{Rc}(\nu,\nu) + \mathrm{div} \nabla_\nu \nu - \mathrm{tr} A^2 \end{aligned}$$

Now note that for X tangent to the surface we have

$$\begin{aligned} \langle \nabla_\nu \nu, X \rangle &= \langle -\nabla_\nu X, \nu \rangle \\ &= \langle [X,\nu], \nu \rangle - \frac{1}{2}\nabla_X \langle \nu,\nu \rangle \\ &= \left\langle \left[X, \frac{1}{v}\partial_t\right], \nu \right\rangle \\ &= \nabla_X \left(\frac{1}{v}\right) \langle \partial_t, \nu \rangle \\ &= -\frac{1}{v}\nabla_X v = \left\langle -\frac{1}{v}\nabla v, X \right\rangle; \end{aligned}$$

and $\langle \nabla_\nu \nu, \nu \rangle = \frac{1}{2}\nabla_\nu \langle \nu,\nu \rangle = 0$, so writing D for the induced connection on the surface we have $\nabla_\nu \nu = D_\nu \nu = -\frac{1}{v} Dv$. Thus

$$\mathrm{div} \nabla_\nu \nu = \langle \nabla_\nu \nabla_\nu \nu, \nu \rangle + \mathrm{div}_N \left(-\frac{1}{v} Dv\right).$$

We have

$$\begin{aligned} \mathrm{div}_N \left(-\frac{1}{v} Dv\right) &= -\frac{1}{v}\Delta_N v - \left\langle D\left(\frac{1}{v}\right), Dv \right\rangle \\ &= -\frac{1}{v}\Delta_N v + \frac{|Dv|^2}{v^2}. \end{aligned}$$

But also

$$\begin{aligned} \langle \nabla_\nu \nabla_\nu \nu, \nu \rangle &= -\left|\nabla_\nu \nu\right|^2 \\ &= -\left|-\frac{1}{v} Dv\right|^2 \\ &= -\frac{|Dv|^2}{v^2}. \end{aligned}$$

Putting these together gives the result

$$\frac{1}{v}\frac{\partial H}{\partial t} = -\mathrm{Rc}\left(\nu,\nu\right) - |A|^2 - \frac{1}{v}\Delta_N v,$$

which is in the case of IMCF

$$\frac{\partial H}{\partial t} = -\Delta_N\left(\frac{1}{H}\right) - \frac{\mathrm{Rc}\left(\nu,\nu\right)}{H} - \frac{|A|^2}{H}. \tag{2.3}$$

2.3 Hawking Mass Monotonicity

Definition 2.3. The Hawking quasi-local mass (or just "Hawking mass") of a 2-surface $N \subset M$ is

$$m_H\left(N\right) := \frac{R\left(N\right)}{2}\left(1 - \frac{1}{16\pi}\int_N H^2 dA\right) \tag{2.4}$$

where H is the mean curvature scalar of N and $R\left(N\right)$ is the "radius" of N; which we define by analogy with the Euclidean sphere as $R\left(N\right) = \sqrt{A\left(N\right)/4\pi}$.

Example 2.4. Spheres in Schwarzschild. Consider a slice $\{t = t_0\}$ of the Schwarzschild spacetime in isotropic coordinates $\{\rho, \theta, \varphi\}$ where the metric has the form

$$g = \left(1 + \frac{m}{2\rho}\right)^4 \delta = \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2\right).$$

Note that the ρ appearing here is not the usual radial coordinate r, but is related by $r = \rho\left(1 + m/2\rho\right)^2$; the horizon of the black hole appears at $\rho = m/2$ and $\rho \sim r$ as $\rho \to \infty$ (so we have the same asymptotic behaviour). To compute the mean curvature of the spheres $S_R = \{\rho = R\}$, we will use Proposition B.1 in a similar fashion to our derivation of (2.2): since the geometry is spherically symmetric, we have

$$\frac{d}{d\rho}\int_{S_\rho} dA = H\left|\partial_\rho\right|\int_{S_\rho} dA$$

from which H can be calculated. The area of a sphere is

$$A_\rho := \int_{S_\rho} dA = \int_{S^2}\left(1 + \frac{m}{2\rho}\right)^4 \rho^2 = 4\pi\rho^2\left(1 + \frac{m}{2\rho}\right)^4$$

and the length of ∂_ρ is

$$\left|\partial_\rho\right| = \sqrt{\left|g_{\rho\rho}\right|} = \left(1 + \frac{m}{2\rho}\right)^2;$$

so we have

$$H\Big|_\rho = \frac{1}{\left|\partial_\rho\right|}\frac{dA_\rho/d\rho}{A_\rho} = 8\rho\left(2\rho + m\right)^{-3}\left(2\rho - m\right).$$

Substituting this into (2.4) gives

$$
\begin{aligned}
m_H(S_\rho) &= \sqrt{\frac{A_\rho}{16\pi}}\left(1-\frac{1}{16\pi}\int_{S_\rho}64\rho^2(2\rho+m)^{-6}(2\rho-m)^2\right)\\
&= \frac{1}{2}\rho(1+m/2\rho)^2\left(1-4\rho^2(2\rho+m)^{-6}(2\rho-m)^2 4\rho^2\left(1+\frac{m}{2\rho}\right)^4\right)\\
&= \frac{1}{8\rho}\left((2\rho+m)^2-(2\rho-m)^2\right)\\
&= m.
\end{aligned}
$$

This is consistent with the idea of measuring the "total mass" enclosed by the surface S_ρ - the Schwarzschild manifold models a spacetime that is vacuum except for a "point mass" m. (Note, however, that in general spacetimes m_H is not a good notion of enclosed mass - for instance, additivity $m_H(\partial A\cup\partial B)=m_H(\partial A)+m_H(\partial B)$ for disjoint A, B *does not hold*.)

For a minimal surface N, we have $H=0$ and therefore $m_H(N)=\sqrt{\mu(N)/16\pi}$: exactly the term appearing on the RHS of Equation 1.4; and as $r\to\infty$, the Hawking mass of the sphere ∂B_r approaches the ADM mass (see Proposition 2.6). The Hawking mass therefore provides a connection between the two quantities appearing in the Penrose inequality, and is the centre of the proof of Theorem 1.12.

Proposition 2.5. *The Hawking mass is non-decreasing under the smooth inverse mean curvature flow of connected surfaces.*

Proof. Let $R(N_t)=\sqrt{A(N_t)/4\pi}$. We found earlier that $L_X dA=dA$ and therefore

$$\frac{d}{dt}A(N_t)=A(N_t)\implies\frac{d}{dt}R(N_t)=\frac{1}{2}R(N_t)$$

and also

$$\frac{\partial H}{\partial t}=-\Delta\left(\frac{1}{H}\right)-\frac{\mathrm{Rc}(\nu,\nu)}{H}-\frac{|A|^2}{H}$$

where D,Δ are the derivatives on the surface. Now we compute

$$
\begin{aligned}
\frac{d}{dt}\int_{N_t}H^2dA &= \int_{N_t}L_X\left(H^2dA\right)\\
&= \int_{N_t}\frac{\partial H^2}{\partial t}dA+\int_{N_t}H^2L_XdA\\
&= \int_{N_t}\left(2H\frac{\partial H}{\partial t}+H^2\right)dA\\
&= \int_{N_t}\left(-2H\Delta\left(\frac{1}{H}\right)-2\mathrm{Rc}(\nu,\nu)+H^2-2|A|^2\right)dA.
\end{aligned}
$$

Taking traces in the Gauss equation for $N\hookrightarrow M$ gives

$$R-2\mathrm{Rc}(\nu,\nu)=R_N+|A|^2-H^2=2\kappa+|A|^2-H^2$$

where κ is the Gaussian curvature of N. This gives us

$$\frac{d}{dt}\int_{N_t}H^2dA=\int_{N_t}\left(-2H\Delta\left(\frac{1}{H}\right)+2\kappa-|A|^2-R\right).$$

Now note that

$$\begin{aligned}
|A|^2 &= \lambda_1^2 + \lambda_2^2 \\
&= \frac{1}{2}(\lambda_1 - \lambda_2)^2 + \frac{1}{2}(\lambda_1 + \lambda_2)^2 \\
&= \frac{1}{2}(\lambda_1 - \lambda_2)^2 + \frac{1}{2}H^2;
\end{aligned}$$

so we can integrate by parts to get

$$\begin{aligned}
\frac{d}{dt}\int_{N_t} H^2 dA &= \int_{N_t}\left(-2H\Delta\left(\frac{1}{H}\right) + 2\kappa - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \frac{1}{2}H^2 - R\right) \\
&= 2\int_{N_t}\kappa - \frac{1}{2}\int_{N_t} H^2 - \int_{N_t}\left(2\frac{|DH|^2}{H^2} + \frac{1}{2}(\lambda_1 - \lambda_2)^2 + R\right) \\
&\leq 4\pi\chi(N_t) - \frac{1}{2}\int_{N_t} H^2
\end{aligned}$$

where we used the Gauss-Bonnet formula and the non-negative scalar curvature of M. For connected N_t we have $\chi(N_t) \leq 2$ and therefore

$$\frac{d}{dt}\int_{N_t} H^2 dA \leq 8\pi\left(1 - \frac{1}{16\pi}\int_{N_t} H^2\right).$$

This implies

$$\begin{aligned}
\frac{d}{dt}m_H(N_t) &= \frac{1}{2}\frac{dR(N_t)}{dt}\left(1 - \frac{1}{16\pi}\int_{N_t} H^2\right) + \frac{R(N_t)}{2}\frac{d}{dt}\left(1 - \frac{1}{16\pi}\int_{N_t} H^2\right) \\
&\geq \frac{1}{2}m_H(N_t) - \frac{1}{2}m_H(N_t) \\
&= 0;
\end{aligned}$$

so $t \mapsto m_H(N_t)$ is non-decreasing. $\square$

2.4 Hawking Mass Asymptotics

Proposition 2.6. *For asymptotically flat spacetime with ADM mass m we have*

$$\lim_{r\to\infty} m_H(\partial B_r) = m$$

where ∂B_r are the coordinate spheres in the asymptotically Euclidean coordinates.

Proof. We will prove this for the case where the metric is conformally flat to first order; i.e. $g = (1 + m/2r)^4\,\delta + O(r^{-2})$. Changing to the associated spherical coordinates (see Appendix A), the metric takes the form

$$g = \begin{pmatrix} (1+m/2r)^4 & & \\ & r^2\sin^2\theta\,(1+m/2r)^4 & \\ & & r^2(1+m/2r)^4 \end{pmatrix} + \begin{pmatrix} O(r^{-2}) & O(r^{-1}) & O(r^{-1}) \\ O(r^{-1}) & O(1) & O(1) \\ O(r^{-1}) & O(1) & O(1) \end{pmatrix}.$$

The area form induced on ∂B_r by g is

$$dA = \left(r^2\sin\theta\,(1+m/2r)^4 + O(1)\right) d\theta \wedge d\varphi;$$

so the area of ∂B_r is

$$\begin{aligned} A_r &= \int_{\partial B_r} \left(r^2 \sin\theta \left(1 + m/2r\right)^4 + O\left(1\right) \right) d\theta d\varphi \\ &= r^2 \left(1 + m/2r\right)^4 \int_{\partial B_r} \sin\theta d\theta d\varphi + \int_{\partial B_r} O\left(1\right) d\theta d\varphi \\ &= 4\pi r^2 \left(1 + m/2r\right)^4 + O\left(1\right) \end{aligned}$$

since θ, φ range over the domain $(0, \pi) \times (0, 2\pi)$ for every r; so if the $O(1)$ term is bounded by C then its integral is bounded by $2\pi^2 C$. Applying the Gram-Schmidt algorithm to the frame $\{\partial_\theta, \partial_\varphi, \partial_r\}$ gives

$$\nu = \left(1 - m/r + O\left(r^{-2}\right)\right) \partial_r + O\left(r^{-3}\right) \partial_\theta + O\left(r^{-3}\right) \partial_\varphi .$$

Computing the Christoffel symbols in the original asymptotically Euclidean frame gives (see Appendix C)

$$H = \partial_i \nu^i + \Gamma^i_{ij} \nu^j = \frac{2}{r} - \frac{4m}{r^2} + O\left(r^{-3}\right).$$

We can now compute

$$\begin{aligned} m_H\left(\partial B_r\right) &= \sqrt{\frac{A_r}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\partial B_r} H^2 dA\right) \\ &= \frac{1}{2} \sqrt{r^2 \left(1 + m/2r\right)^4 + O\left(1\right)} \left(\frac{2m}{r} + O\left(r^{-2}\right)\right). \end{aligned}$$

When we take the limit $r \to \infty$ we can drop all but the highest order (in r) terms in both multiplicands, giving

$$\lim_{r \to \infty} m_H\left(\partial B_r\right) = \lim_{r \to \infty} \frac{1}{2} \sqrt{r^2} \frac{2m}{r} = m.$$

□

Thus what needs to be shown is that the surfaces N_t converge to the large coordinate spheres in some strong enough sense to control the Hawking mass. Intuitively, we expect this to be true - any deformation from a sphere is naturally smoothed out by the flow, since any "flattening" has less curvature and thus will expand faster than the rest of the surface, thus catching up, and similarly sharp extrusions have a lot of curvature and would fall back. We will see this is indeed the case (in a more general form) in Section 3.

2.5 Rigidity

We can now prove the rigidity portion of Theorem 1.12 assuming the existence of a smooth inverse mean curvature flow.

Proposition 2.7. *If $m_{ADM} = \sqrt{A\left(N_0\right)/16\pi}$ and M admits a smooth IMCF solution of connected surfaces starting at N_0 and asymptotic to large spheres, then (M, g) is isometric to the Schwarzschild slice.*

Proof. The equality implies that $m_H\left(N_t\right) = m_{ADM}$ for all t, and thus that $dm_H\left(N_t\right)/dt = 0$ for all t; i.e. the terms discarded in the computations of Proposition 2.5 are in fact all zero. Firstly we see $\chi\left(N_t\right) = 2$, so the N_t are topologically 2-spheres. This implies $R = 0$ everywhere, and on each N_t we have $DH = 0$ and $\lambda_1 = \lambda_2$; so the surfaces have constant principal curvatures $\lambda_1 = \lambda_2 = H(t)/2$. By the evolution equation (2.3), $\mathrm{Rc}(\nu, \nu)$ is also constant on N_t, and thus the Gaussian curvature $\kappa = \lambda_1\lambda_2 - \mathrm{Rc}(\nu, \nu)$ is constant on N_t; so N_t is isometric to a round sphere. Since the concentric spheres N_t foliate M, we can write the metric on M using the flow time coordinate along with those on the 2-sphere, which gives (noting that ∂_t has length $v = H^{-1}$)

$$g = H^{-2}dt^2 + d\Omega_t^2$$

where $d\Omega_t^2$ is the round metric on S^2 scaled to have total area $A(N_t)$; i.e.

$$d\Omega_t^2 = \frac{A(N_t)}{4\pi}d\Omega^2 = \frac{A(N_0)}{4\pi}e^t d\Omega^2.$$

We now make a change of variables $r = e^{t/2}\sqrt{\frac{A(N_0)}{4\pi}}$ so that $A\left(N_{t(r)}\right) = 4\pi r^2$, which gives

$$\begin{aligned} g &= H^{-2}d\left(\ln\left(\frac{4\pi r^2}{A(N_0)}\right)\right)^2 + r^2 d\Omega^2 \\ &= \frac{4H^{-2}}{r^2}dr^2 + r^2 d\Omega^2. \end{aligned}$$

Since $m_H(N_t) = m_{\mathrm{ADM}}$, we have

$$m_{\mathrm{ADM}} = \frac{r}{2}\left(1 - \frac{1}{16\pi}\int_{N_{t(r)}} H^2 dA\right)$$

and thus (since H is constant on N_t)

$$\begin{aligned} H^2 &= \frac{16\pi}{A\left(N_{t(r)}\right)}\left(1 - \frac{2m_{\mathrm{ADM}}}{r}\right) \\ &= \frac{4}{r^2}\left(1 - \frac{2m_{\mathrm{ADM}}}{r}\right). \end{aligned}$$

Thus we have

$$g = \left(1 - \frac{2m_{\mathrm{ADM}}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

the familiar Schwarzschild metric.

□

3 Weak Inverse Mean Curvature Flow

3.1 Motivation

We have established the Riemannian Penrose Inequality for the case where there exists a smooth geometric flow solving (2.1) such that $E_t \to S_r$ with $r \to \infty$ as $t \to \infty$. However, it is not hard at all to see that such a solution is not guaranteed.

Example 3.1. Consider a thin torus in $\mathbb{R}^3$ parametrised by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta\,(1+\epsilon\cos\varphi) \\ \sin\theta\,(1+\epsilon\cos\varphi) \\ \epsilon\sin\varphi \end{pmatrix}.$$

This has mean curvature

$$H = \frac{1+2\epsilon\cos\varphi}{\epsilon\,(1+\epsilon\cos\varphi)}.$$

For $\epsilon \ll 1$ this is $\epsilon^{-1} + \cos\varphi + O(\epsilon) \approx \epsilon^{-1} > 0$, so using it as an initial condition for (2.1), the flow can initially be approximated by exponential growth of the tube radius ϵ. However, as ϵ increases, the mean curvature will approach 0 on the inner edge of the (distorted) torus ($\cos\varphi = -1$); so there is a singularity in the velocity and we cannot continue it.

Thus we must generalise our definition of the flow (i.e. move to a weak reformulation) if we want global existence for general initial conditions and geometries. As the first step towards a weak formulation, we recast (2.1) in terms of the level sets of some function:

Let $u : M \to \mathbb{R}$ be a function (which one can think of as the "time" in the flow) and let the surface at time t be $N_t := \partial\{u < t\}$. Then if u is differentiable and $|\nabla u| > 0$, one sees that the normal velocity is $X = \nabla u / |\nabla u|^2$; so since

$$H = \operatorname{div}_N\left(\frac{\nabla u}{|\nabla u|}\right),$$

we find that for regular points of u, Equation 2.1 is equivalent to

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|\,. \tag{3.1}$$

One advantage of the level-set formulation (3.1) over the geometric flow equation (2.1) is that it (at least partially) provides a mechanism to avoid the singularities - if we can devise a scheme to have the surface jump "over" the singularities (in the example to jump from a torus to some topologically spherical hull), then this can be represented as a plateau of the function u. Of course this will require significant changes to the formulation, as (3.1) is not defined for $\nabla u = 0$.

Equation (3.1) is a degenerate elliptic PDE. The program of Huisken and Ilmanen was to define weak solutions of Equation 3.1, prove the global existence of these solutions in the asymptotically flat case and show the required results on m_H still hold for weak solutions.

3.2 Variational Formulation

Note. *Throughout this section we will introduce the weak formulation of the IMCF, where we must drop assumptions of smoothness of surfaces. For simplicity we will use notation consistent with the previous discussion; but note that the actual definitions are more subtle. The changes are: V, dV now refer to the n-dimensional Lebesgue measure, A, dA to the $n-1$-dimensional Hausdorff measure, H to the weak mean curvature and ∂ to the reduced boundary. See Appendix D for details.*

The variational approach to weak reformulation of second-order PDEs on a compact domain K is to find a Lagrangian $\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} : (u, \nabla u) \mapsto \mathcal{L}(u, \nabla u)$ such that the Euler-Lagrange equations

$$\operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} = \frac{\partial \mathcal{L}}{\partial u} \tag{3.2}$$

are equivalent to the original PDE. One then defines the *action functional*

$$S_{\mathcal{L}}^K : \mathcal{A} \to \mathbb{R} : u \mapsto \int_K \mathcal{L}(u(x), \nabla u(x))\, dx$$

for some convenient space of functions $\mathcal{A} \subset \mathbb{R}^K$. A weak solution is then defined as a local minimiser of $S_{\mathcal{L}}^K$, and one finds that every weak solution that is C^2 is in fact a strong solution to the original PDE. The existence of weak solutions is usually proven by endowing $\mathcal{A}$ with additional structure (usually at least Banach) and applying the theory of functional analysis. There are then various cases (e.g. linear elliptic PDEs) where the weak solutions are in fact guaranteed to be strong solutions; this is *regularity theory.* (For example, a classic result is that every weakly harmonic function is in fact analytic.) For our purposes, we will use $\mathcal{A} = C^{0,1}_{\text{loc}}(\Omega)$ where Ω is the domain of interest; i.e. the locally Lipschitz continuous functions, which (when restricted to a precompact open subset Ω' so $\mathcal{A}' = C^{0,1}(\Omega')$) forms a Banach space when given the norm

$$\|u\| = \sup |u| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x, y)}.$$

We will follow the weak formulation of the IMCF and subsequent proof of the Penrose inequality given by Huisken and Ilmanen in [1]. Equation 3.1 does not appear to have the form of the Euler-Lagrange equation; so we instead "freeze" the $|\nabla u|$ on the RHS; i.e. for a given $u \in \mathcal{A}$, define the Lagrangian

$$\mathcal{L}_u(v) = v |\nabla u| + |\nabla v|, \tag{3.3}$$

which has Euler-Lagrange equation

$$\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = |\nabla u|. \tag{3.4}$$

We can now recover Equation 3.1 by setting $u = v$. At first this seems strange, but we will see that we still have a minimisation principle. Call the action $J_u^K(v) := \int_K \mathcal{L}_u(v, \nabla v)$ for compact sets K.

Definition 3.2. A locally Lipschitz function u is a weak solution of the IMCF (or a "WIMCF solution") if for every locally Lipschitz v with $\{u \neq v\} \subset\subset K$ we have $J_u^K(u) \leq J_u^K(v)$. u is a *subsolution* or *supersolution* if this holds for $v \leq u$, $v \geq u$ respectively.

If u is C^2, this implies u satisfies Equation 3.4 with $u = v$; i.e. u is a strong solution; so we still have a useful weak formulation of the problem. We restrict ourselves to compact sets because we expect the function u (and therefore its competitors v) to grow unbounded as $r \to \infty$ (indeed this is necessary to conclude the main theorem!), so assumptions of global integrability are too strong. The minimisation principle does not depend on the choice of K, as long as it contains the support of $u-v$; thus we will often omit the K.

It is also simple to impose initial conditions:

Definition 3.3. A WIMCF solution u satisfies the initial condition given by a surface $N_0 = \partial E_0 \subset M$ if $E_0 = \{u < 0\}$.

Since the level sets play such a central role, it becomes convenient to cast the problem in terms of the sub-level sets $E_t = \{u < t\}$, using the new functional

$$\mathcal{J}_u^K(E) = A(\partial E \cap K) - \int_{E \cap K} |\nabla u| \,.$$

This functional is lower semi-continuous with respect to convergence of sets in measure; i.e. if $V(E_n \Delta E) \to 0$ then $\mathcal{J}_u(E) \leq \liminf_{n\to\infty} \mathcal{J}_u(E_n)$. (This is a straightforward consequence of Prop D.7.) It also satisfies the inequality

$$\mathcal{J}_u(E \cap F) + \mathcal{J}_u(E \cup F) \leq \mathcal{J}_u(E) + \mathcal{J}_u(F) \tag{3.5}$$

since the same inequality holds for the surface area and equality holds for the $\int |\nabla u|$ term. We will often omit the intersections with K.

The following proposition recasts the solution criteria in terms of $\mathcal{J}_u$.

Proposition 3.4. *u minimises J_u amongst v satisfying $\{u \neq v\} \subset\subset K \iff$ for each t, E_t minimises $\mathcal{J}_u$ amongst all sets F satisfying $F \Delta E_t \subset\subset K$.*

Proof. ($\Longleftarrow$) For $u, v \in C^{0,1}_{\text{loc}}(\Omega)$ define $E_t = \{u < t\}$, $F_t = \{v < t\}$, and let (a,b) be a bounding interval for the image of u and v on K (guaranteed by compactness of K and continuity of u, v). Then we have by the co-area formula (see Appendix D)

$$\begin{aligned} J_u(v) &= \int_K v\,|\nabla u| + \int_K |\nabla v| \\ &= \int_K v\,|\nabla u| + \int_a^b A(\partial F_t)\,dt \end{aligned}$$

where we first restricted the second integral to $K \backslash \{|\nabla v| = 0\}$, where $\{v = t\} = \partial\{v < t\}$. Now write

$$\begin{aligned} v &= -(b-v) + b \\ &= -\int_a^b \chi_{\{t>v\}}\,dt + b. \end{aligned}$$

This yields

$$J_u(v) = \int_a^b \left(A(\partial F_t) - \int_K \chi_{\{t>v\}}\,|\nabla u| \right) dt + b \int_K |\nabla u| \,. \tag{3.6}$$

But $\{v < t\} \cap K$ is exactly $F_t \cap K$, so we recognise $\mathcal{J}_u(F_t)$:

$$J_u(v) = \int_a^b \mathcal{J}_u(F_t)\,dt + b\int_K |\nabla u| \,.$$

If $\mathcal{J}_u(E_t) \leq \mathcal{J}_u(\{v < t\})$ for every competitor v, then we have $\int_a^b \mathcal{J}_u(E_t) \leq \int_a^b \mathcal{J}_u(F_t)$ and therefore $J_u(u) \leq J_u(v)$; i.e. u minimises J_u.

($\Longrightarrow$) The other direction is more subtle - we will show that if u is a supersolution then each E_t minimises $\mathcal{J}_u$ on the inside (with respect to competitors $F \subset E_t$), and likewise for supersolutions E_t minimises $\mathcal{J}_u$ on the outside (with respect to competitors $F \supset E_t$).

First, let u be a supersolution, and choose any $F \subset E_{t_0}$ for any time t_0 such that $F \Delta E_{t_0}$ is compact. Order the collection

$$\left\{ A \text{ is a finite-perimeter subset of } E_{t_0} \middle| F \subset A, \mathcal{J}_u(A) \leq \mathcal{J}(F) \right\}$$

by the relation $A \lhd B$ if and only if $A \subset B$ and $\mathcal{J}_u(A) > \mathcal{J}_u(B)$. For any chain A_n, we have by lower semi-continuity of $\mathcal{J}_u$

$$\mathcal{J}_u\left(\bigcup_n A_n\right) \leq \liminf_n \mathcal{J}_u(A_n) \leq \mathcal{J}_u(A_j)\, \forall j$$

so $\bigcup_n A_n$ is an upper bound for the chain. By Zorn's Lemma we therefore have a maximal element; i.e. an $F' \subset E_{t_0}$ such that $\mathcal{J}_u(G) \geq \mathcal{J}_u(F')$ whenever $G \supset F'$. Now define

$$v = \begin{cases} t_0 & \text{on } E_{t_0} \backslash F' \\ u & \text{elsewhere} \end{cases}$$

so that

$$F_t = \begin{cases} E_t & t > t_0 \\ E_t \cap F' & t \leq t_0 \end{cases}.$$

While v is not $C^{0,1}_{\text{loc}}$ and thus we cannot immediately use it as a competitor $v \geq u$, it is locally bounded and of locally bounded variation; so $J_u(v)$ is well defined (if we interpret $\int |\nabla v|$ as the total variation). Approximating it with smooth functions $v_i \to v$ on K such that $|\nabla v_i| \rightharpoonup |\nabla v|$ in $\mathcal{D}'$ weak-* and applying the fact that u is a supersolution gives $J_u(v) \geq J_u(u)$ and therefore (using Equation 3.6)

$$\int_a^b \mathcal{J}_u(F_t)\, dt \geq \int_a^b \mathcal{J}_u(E_t)\, dt.$$

For $t \leq t_0$ we have (using (3.5) and the maximality of F')

$$\mathcal{J}_u(F_t) + \mathcal{J}_u(F') \leq \mathcal{J}_u(F_t) + \mathcal{J}_u(E_t \cup F') \leq \mathcal{J}_u(E_t) + \mathcal{J}_u(F') \tag{3.7}$$

and therefore

$$\mathcal{J}_u(F_t) \leq \mathcal{J}_u(E_t).$$

(We obviously have equality for $t > t_0$.) Thus we must in fact have $\mathcal{J}_u(F_t) = \mathcal{J}_u(E_t)$ for a.e. $t \in [a, b]$. Making this substitution in (3.7) gives

$$\mathcal{J}_u(E_t \cup F') \leq \mathcal{J}_u(F')$$

for almost every t; so taking the limit $t \nearrow t_0$ and applying lower semicontinuity gives

$$\mathcal{J}_u(E_{t_0}) \leq \mathcal{J}_u(F') \leq \mathcal{J}_u(F);$$

i.e. E_{t_0} minimises $\mathcal{J}_u$ on the inside.

Now assume u is a subsolution and consider any $F \supset E^+_{t_0} = \{u \leq t_0\}$ for any t_0. Then using the exact same argument as before (just flipping the direction of set inclusion in the definition of $\triangleleft$) we can pass to an $F' \subset F$ with $\mathcal{J}_u(F') \leq \mathcal{J}_u(F)$ such that for every G satisfying $E_{t_0}^+ \subset G \subset F'$ we have $\mathcal{J}_u(G) \geq \mathcal{J}_u(F')$. Defining v so that

$$F_t = \begin{cases} E_t & t < t_0 \\ E_t \cup F' & t \geq t_0 \end{cases}$$

and using the same argument based on (3.6) gives $\mathcal{J}_u(F_t) = \mathcal{J}_u(E_t)$ for a.e. t, so we have $\mathcal{J}_u(E_t \cap F') \leq \mathcal{J}_u(F')$. Passing to the limit $t \searrow t_0$ using the lower semicontinuity gives

$$\mathcal{J}_u\left(E_{t_0}^+\right) \leq \mathcal{J}_u(F') \leq \mathcal{J}_u(F);$$

i.e. $E_{t_0}^+ = \{u \leq t_0\} = \bigcap_{t > t_0} E_t$ minimises $\mathcal{J}_u$ on the outside. Since $E_t^+ \to E_{t_0}$ locally as $t \nearrow t_0$, we have $\mathcal{J}_u(E_{t_0}) \leq \liminf_{t \nearrow t_0} \mathcal{J}_u\left(E_t^+\right)$. Given an $F \supset E_{t_0}$, we have $F \supset E_t^+$ for all $t < t_0$ and therefore $\mathcal{J}_u\left(E_t^+\right) \leq \mathcal{J}_u(F)$; so $\mathcal{J}_u(E_{t_0}) \leq \mathcal{J}_u(F)$; i.e. E_{t_0} itself minimises $\mathcal{J}_u$ on the outside.

Thus if u minimises J_u amongst all competitors v then each E_t minimises $\mathcal{J}_u$ amongst all competitors F. □

The relationship between $\mathcal{J}_u$ and surface area means that the following definition will come in useful when determining the geometric consequences of the minimisation principle - in particular, the theory of minimising hulls will allow us to investigate the behaviour of the flow at the jumps.

Definition 3.5. A set $E \subset M$ is called a *minimising hull* if it minimises surface area on the outside; i.e. if $A(\partial E) \leq A(\partial F)$ whenever $F \supset E$. It is called a *strictly minimising hull* if equality holds only for $F = E$ a.e.

For a given set E, there is a unique smallest strictly minimising hull E' containing E given by the intersection of all such strictly minimising hulls. Call E' *the strictly minimising hull of* E.

When ∂E is C^2, $\partial E'$ is $C^{1,1}$ and $\partial E' \backslash \partial E$ is C^∞ [1]. Therefore any smooth variation of $\partial E'$ supported inside $\partial E' \backslash \partial E$ will stay inside the minimisation domain (i.e. be the boundary of a set $F \supset E$) for small enough parameter, giving $H = 0$ on $\partial E' \backslash \partial E$.

Proposition 3.6. *For a weak IMCF solution u:*

1. each $E_t = \{u < t\}$ is a minimising hull;
2. each $E_t^+ = \{u \leq t\}$ is a strictly minimising hull for $t > 0$;
3. $E_t' = E_t^+$; and
4. $A(\partial E_t) = A\left(\partial E_t^+\right)$.

Proof.

1. Since E_t minimises $\mathcal{J}_u$, we have

$$A(\partial E_t) - \int_{E_t} |\nabla u| \leq A(\partial F) - \int_F |\nabla u|$$

and therefore when $F \supset E_t$

$$A(\partial E_t) \leq A(\partial F).$$

2. We saw in the proof of Prop 3.4 that E_t^+ also minimises $\mathcal{J}_u$, so the same proof works. To show it is strict, assume $F \supset E_t^+$ and $A(\partial F) = A\left(\partial E_t^+\right)$. Then we must have

$$\int_{F \backslash E_t^+} |\nabla u| = 0,$$

thus u is constant on each connected component of $F\backslash E_t^+$. Since M is connected, each component of $F\backslash E_t^+$ has closure touching $\{u = t\}$, so by continuity of u, $u = t$ on $F\backslash E_t^+$. By the definition of E_t^+ this implies $F \subset E_t^+$, so $F = E_t^+$.

3. Since $E_t^+ \supset E_t$ is a strictly minimising hull, we have $E_t^+ \supset E_t'$ by definition; so $\int_{E_t^+} |\nabla u| = \int_{E_t'} |\nabla u|$ since $E_t^+\backslash E_t'$ lies in the level set $\{u = t\}$, where $|\nabla u| = 0$ almost everywhere. This implies $A\left(\partial E_t^+\right) \leq A(\partial E_t')$ (since E_t^+ minimises $\mathcal{J}_u$) and therefore $E_t^+ = E_t'$ since E_t' is strictly minimising.

4. Both E_t and E_t^+ minimise $\mathcal{J}_u$ so $\mathcal{J}_u(E_t) = \mathcal{J}_u\left(E_t^+\right)$; and $\int_{E_t} |\nabla u| = \int_{E_t^+} |\nabla u|$ as in the proof of 3, so this means $A(\partial E_t) = A\left(\partial E_t^+\right)$.

$\square$

With these facts established, we can now easily prove:

Corollary 3.7. *The Geroch monotonicity holds at jumps; i.e.* $m_H\left(E_t^+\right) \geq m_H(E_t)$.

Proof. We have $A(\partial E_t) = A\left(\partial E_t^+\right)$ by Prop 3.6.4. Since $E_t^+ = E_t'$ by Prop 3.6.3, the theory of minimising hulls gives

$$H_{\partial E_t^+} = \begin{cases} 0 & \text{on } \partial E_t^+\backslash\partial E_t \\ H_{\partial E_t} & \text{on } \partial E_t^+ \cap \partial E_t \end{cases}.$$

Thus

$$\begin{aligned} m_H\left(\partial E_t^+\right) &= \sqrt{\frac{A\left(\partial E_t^+\right)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\partial E_t^+} H_{\partial E_t^+}^2\right) \\ &= \sqrt{\frac{A(\partial E_t)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\partial E_t^+ \cap \partial E_t} H_{\partial E_t}^2\right) \\ &\geq \sqrt{\frac{A(\partial E_t)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\partial E_t} H_{\partial E_t}^2\right) = m_H(\partial E_t). \end{aligned}$$

$\square$

We now present a simple example of the weak flow, with an analysis of the jumping phenomenon and the behaviour of the Hawking mass.

Example 3.8. Two spheres. Let M be Euclidean 3-space and consider the initial condition $E_0 = B_{r_0}(-1,0,0) \cup B_{r_0}(1,0,0)$ for some $r_0 \ll 1$; so the initial surface consists of two spheres of radius r_0 with centres a distance 2 apart. The mean curvature of a sphere of radius r is the constant $2r^{-1}$; so we have

$$\begin{aligned} m_H(\partial E_0) &= \sqrt{\frac{A(\partial E_0)}{16\pi}}\left(1 - \frac{2^2}{16\pi r_0^2} A(\partial E_0)\right) \\ &= -\frac{r_0}{2}. \end{aligned}$$

Initially, the flow will proceed as it does for a single sphere: exponential growth of the radius $r = r_0 e^{t/2}$

$$E_t = B_r(-1,0,0) \cup B_r(1,0,0), \ t < t_{\mathrm{jump}}.$$

During this period of smooth flow, the Hawking mass will *decrease* as $m_H(\partial E_t) = -r/2$! This is a reminder of how important the connectedness of N_t is for the Geroch monotonicity to hold. The smooth flow would self-intersect once $r = 1$; but our weak flow must jump before this time, as the two touching spheres have surface area 8π while being contained in a rounded cylinder with area $2(2\pi + 2\pi) = 8\pi$, so a slight pinching of this cylinder would decrease the area and contradict Proposition 3.6.

Let us now attempt to find the actual jump time t_{jump} and the resultant minimising hull $\partial E^+_{t_{\mathrm{jump}}}$. From the rotational symmetry, we know the surface must remain a surface of revolution about $y = z = 0$; and we know that any new part of the surface must be minimal. Thus the new portion of the surface must be a catenoid; so we will end up with a catenoidal bridge between two spheres (something like a dumbbell; see Figure 3). Since the two-spheres configuration also has a mirror symmetry, the catenoid will be centred on the origin. Thanks to the spherical and reflection symmetry, we need only consider a cross-section $z = 0, x > 0$; the spherical and catenoidal surfaces will then be represented in general by the revolutions of the functions

$$\begin{aligned} s_r(x) &= \sqrt{r^2 - (x-1)^2} \\ c_a(x) &= \frac{1}{a}\cosh ax. \end{aligned}$$

For the two surfaces to join together into a C^1 surface (which must occur since the strictly minimising hull is $C^{1,1}$), we need some point $x = x_0$ where

$$s_r(x_0) = c_a(x_0), \ s_r'(x_0) = c_a'(x_0). \tag{3.8}$$

For small r this has no solution (i.e. there is no catenoid spanning the gap between the spheres); but for each r in some interval $[r_0, 1]$ there is a unique solution (x_0, a) that satisfies these constraints. The area gained during the jump is

$$\begin{aligned} \delta A(r) &= \text{Area of catenoidal section} - \text{Area of 2 spherical caps} \\ &= 2\left[2\pi\int_0^{x_0} c_a(x)\sqrt{1 + c_a'(x)^2}dx - 2\pi\int_{1-r}^{x_0} s_r(x)\sqrt{1 + s_r'(x)^2}dx\right]. \end{aligned}$$

The surface will jump at the first time when $\delta A(r) \le 0$; i.e. when the two spheres have area greater than or equal to the dumbbell. The constraints (3.8) are not easily solved analytically; but using a numerical root-finding algorithm (see Appendix C), we can produce approximate values of δA - see Figure 1. Applying root-finding to the function δA then gives the jump radius

$$r_0 e^{t_{\mathrm{jump}}/2} = r_{\mathrm{jump}} \approx 0.862.$$

The surface will gain some Hawking mass during the jump (we know that area is preserved, and the spherical caps we lost had a positive contribution to $\int H^2$), but we still have $m_H\left(\partial E^+_{t_{\mathrm{jump}}}\right) < 0$. As the flow continues out to infinity, the Hawking mass will (by the results of the previous section) increase with limit $m_{\mathrm{ADM}} = 0$.

Figure 2 shows cross-sections of the flow at regular times, with the final surface shown being $E^+_{t_{\mathrm{jump}}}$.

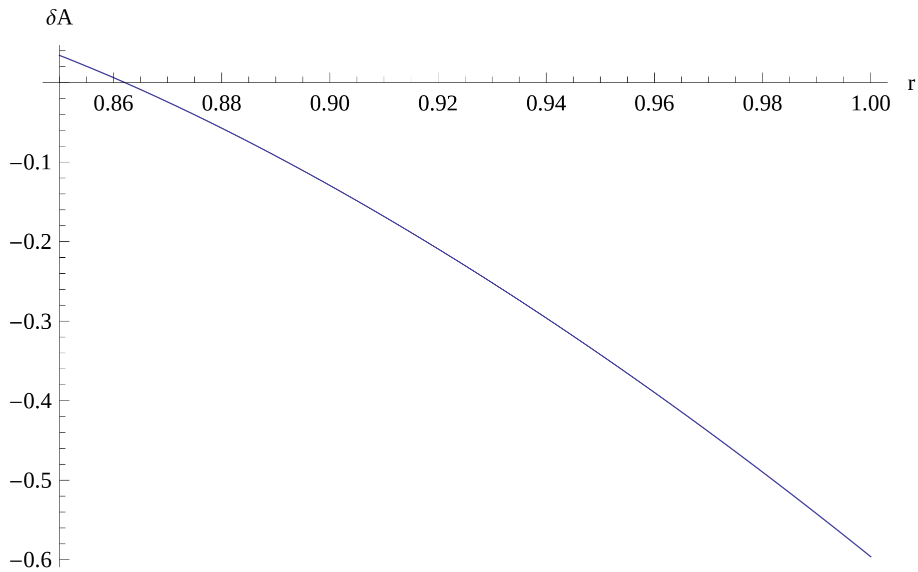

Figure 1: $\delta A\,(r)$ vs r for solutions of (3.8)

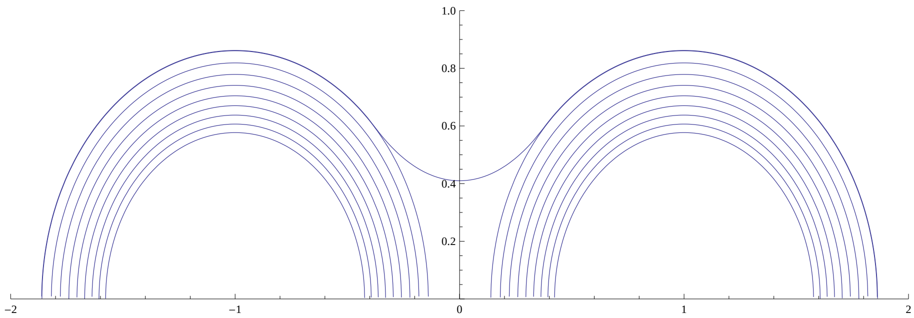

Figure 2: Cross-section of weak inverse mean curvature flow. Initial condition is $r_0 \approx 0.577$.

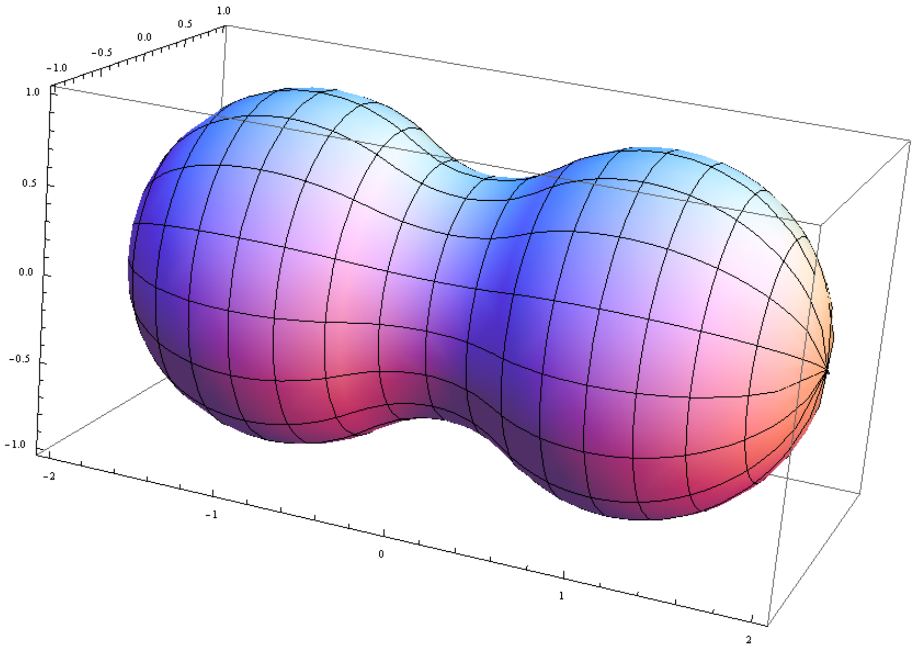

Figure 3: The minimising hull $E^{+}_{t_{\mathrm{jump}}}$.

3.3 Elliptic Regularisation

The existence of weak IMCF solutions is proven using an approximation scheme known as Elliptic Regularisation. We move up a dimension and consider translates of the graph of u in $M \times \mathbb{R}$: the surfaces

$$N_t^\epsilon(x) = \left(x, \frac{u(x) - t}{\epsilon}\right).$$

These are level sets of the function $U : M \times \mathbb{R} \to \mathbb{R}$ given by

$$U(x, \zeta) = \frac{u(x) - \zeta}{\epsilon}$$

and therefore (substituting $u \to U$ in (3.1) and simplifying) N_t^ϵ is an IMCF if and only if

$$E^\epsilon(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon^2}}\right) - \sqrt{|\nabla u|^2 + \epsilon^2} = 0. \tag{3.9}$$

This new equation is non-degenerate elliptic for $\epsilon > 0$ (we no longer have a blow-up when $\nabla u \to 0$). We will show there exist smooth solutions u^ϵ of (3.9) on domains that exhaust M as $\epsilon \to 0$, with u^ϵ remaining locally uniformly bounded. We can then take a convergent subsequence and pass to the limit $\epsilon \to 0$, giving a weak IMCF on $M \times \mathbb{R}$ whose level sets are vertical cylinders (i.e. $N_t = \{x \in M | u^\epsilon(x) \to t\} \times \mathbb{R}$). Intersecting these cylinders with M will give a solution to the original weak IMCF problem. We assume the existence of a smooth weak subsolution v with $\nabla v \neq 0$ everywhere; these always exist in asymptotically flat manifolds (consider functions of the form $C \ln r$ in the asymptotic region). Let $F_t = \{v < t\}$.

Lemma 3.9. *(A priori estimates for regularised solutions). [1, p384] For every $L > 0$ there is an $\epsilon(L) > 0$ such that for all $\epsilon \in (0, \epsilon(L)]$ and all $\tau \in [0, L-2]$: If u solves (3.9) on $\Omega_L = F_L \backslash E_0$ with $u = 0$ on ∂E_0 and $u = \tau$ on ∂F_L then u satisfies the estimates*

$$\begin{aligned}
u &\geq -\epsilon && \text{in } \Omega_L, \\
u &\geq v + \tau - L && \text{in } \bar{F}_L \backslash F_0, \\
|\nabla u| &\leq \max(H, 0) + \epsilon && \text{on } \partial E_0, \\
|\nabla u| &\leq C(L) && \text{on } \partial F_L, \\
\|u\|_{2;\alpha} &\leq C(\epsilon, L) && \text{and} \\
|\nabla u(x)| &\leq \max_{\partial\Omega_L \cap B_r(x)} |\nabla u| + \epsilon + C/r
\end{aligned}$$

for any r such that $B_r(x)$ is diffeomorphic to a Euclidean ball (B_ρ, δ) such that the metric components satisfy $|g_{ij} - \delta_{ij}| \leq 1/100$, $|g_{ij,k}| \leq 1/100r$.

Theorem 3.10. *For any $L > 0$, there is an $\epsilon > 0$ and a solution of (3.9) such that $u = 0$ on ∂E_0 and $u = L - 2$ on $\{v = L\}$.*

Proof. We study the related problem with modified boundary condition $u = \tau$ on $\{v = L\}$, where $0 \leq \tau \leq L - 2$. Define

$$F(w, \epsilon) = F^\epsilon(w) = \operatorname{div}\left(\frac{\nabla w}{\sqrt{1 + |\nabla w|^2}}\right) - \epsilon\sqrt{1 + |\nabla w|^2};$$

then u solves (3.9) if and only if $F^{\epsilon}(u/\epsilon) = 0$. For the case $\tau = 0$, we consider F as a map of Banach spaces

$$F : C_0^{2;\alpha}\left(\bar{\Omega}_L\right) \times \mathbb{R} \to C^{0;\alpha}\left(\bar{\Omega}_L\right)$$

where $\Omega_L = \{v < L\} \backslash E_0$. We have a solution $F(0,0) = 0$, and F is differentiable in the $C_0^{2;\alpha}$ directions at 0:

$$\begin{aligned} d_0 F^0(w) &= \left.\frac{d}{dt}\right|_{t=0} F^0(tw) \\ &= \operatorname{div}\left(\left.\frac{d}{dt}\right|_{t=0} \frac{t\nabla w}{\sqrt{1+t^2|\nabla w|^2}}\right) - \epsilon \left.\frac{d}{dt}\right|_{t=0} \sqrt{1+t^2|\nabla w|^2} \\ &= \Delta w; \end{aligned}$$

i.e. the linearised version of $F^{\epsilon}(w) = 0$ is simply Laplace's equation. Since $\Delta : C_0^{2;\alpha}\left(\bar{\Omega}_L\right) \to C^{0;\alpha}\left(\bar{\Omega}_L\right)$ is an isomorphism (i.e. the Dirichlet problem has a unique Hölder solution for Hölder data), the Implicit Function Theorem for Banach spaces tells us there is an interval $(-\epsilon_0, \epsilon_0)$ and a function $S : (-\epsilon_0, \epsilon_0) \to C_0^{2;\alpha}\left(\bar{\Omega}_L\right)$ such that $F(S(\epsilon), \epsilon) = 0$ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$; i.e. for $\tau = 0$, there is a solution $F^{\epsilon}(u) = 0$ for some $\epsilon > 0$; fix such an ϵ.

We must now extend this to $\tau \neq 0$; i.e. to more general boundary conditions. Let I be the set of values τ such that (3.9) has a solution with $u = \tau$ on ∂F_L. We just showed $0 \in I$. Take solutions u_{τ_j} with $\tau_j \in I \cap [0, L-2]$ an arbitrary sequence converging to τ; then the uniform Hölder estimate in Lemma 3.9 implies that (passing to a subsequence) we have local uniform convergence $u_{\tau_j} \to u$ (since the Arzela-Ascoli theorem implies bounded sets in $C^{0,\alpha}\left(\bar{\Omega}_L\right)$ are compact in $C\left(\bar{\Omega}_L\right)$). Since E^{ϵ} is continuous as a map $C^2\left(\bar{\Omega}_L\right) \to C^0\left(\bar{\Omega}_L\right)$, the limit satisfies $E^{\epsilon}(u) = 0$; and since $u_{\tau_j} = \tau_j$ on the outer boundary we have $u = \tau$ on the outer boundary. Thus $\tau \in I \cap [0, L-2]$, so $I \cap [0, L-2]$ is closed. To show I is open we use the same linearisation method as before, this time expanding about an arbitrary u (not just $u = 0$). Consider the operator

$$\begin{aligned} G^{\tau} &: C^{2,\alpha}\left(\bar{\Omega}_L\right) \to C^{0,\alpha}\left(\bar{\Omega}_L\right) \times C^{2,\alpha}\left(\partial\Omega_L\right) \\ G(u,\tau) &= G^{\tau}(u) := \left(E^{\epsilon}(u), u|_{\partial\Omega_L} - \tau\chi_{\partial F_L}\right) \end{aligned}$$

so that u is a solution with outer boundary value τ if and only if $G^{\tau}(u) = 0$. Clearly $dG^{\tau}|_u(w) = \left(dE^{\epsilon}|_u(w), w|_{\partial\Omega_L}\right)$. Since $E^{\epsilon}(u)$ depends only on ∇u, the linearisation $dE^{\epsilon}|_u(w) = 0$ is of the form

$$\operatorname{div}(M(x) \cdot \nabla v) + V(x) \cdot \nabla v = 0$$

for some matrices M and vectors V; so the maximum principle for linear elliptic PDEs implies that the linearisation $dE(w) = 0$ has only solution $w = 0$; i.e. dE is injective. The existence and regularity theory for elliptic PDEs guarantees solutions for the linearised equation, so dE is an isomorphism. Applying the Implicit Function Theorem as before gives solutions with outer boundary values in a neighbourhood about τ; i.e. I is open. Thus $I \cap [0, L-2]$ is both closed and open when viewed as a subset of the space $[0, L-2]$ (with the subspace topology from $\mathbb{R}$), so $I \supset [0, L-2]$. Thus we have a solution with the desired boundary condition $u = L-2$ on ∂F_L. $\square$

Now that we have solutions to the regularised equations, we need to show that we can take a limit and obtain a weak IMCF solution on M.

Theorem 3.11. *If u_i are weak IMCF solutions on open sets Ω_i such that $u_i \to u$ and $\Omega_i \to \Omega$ locally uniformly and $\sup_K |\nabla u_i|$ is eventually bounded for each $K \subset\subset \Omega$, then u is a weak IMCF* *solution on Ω.[1, p375]*

Theorem 3.12. *For any precompact smooth open set E_0, there exists a locally Lipschitz weak IMCF solution with initial condition E_0 (assuming the existence of a subsolution v as before).*

Proof. Take a sequence $L_j \to \infty$ and a corresponding sequence $\epsilon_j \to 0$ such that for each j, we have a solution u_j of (3.9) with outer boundary condition $u_j|_{\partial F_L} = L-2$. Combining the estimates for $|\nabla u|$, we find that for large enough j (so that $B_r(x)$ does not intersect ∂F_{L_j})

$$|\nabla u_j(x)| \le \max_{\partial E_0 \cap B_r(x)} \max(H,0) + 2\epsilon_j + C/r \le \max_{\partial E_0 \cap B_r(x)} \max(H,0) + C + C/r$$

i.e. the u_j are equicontinuous on compact subsets. Thus by Arzela-Ascoli we can pass to a subsequence and obtain local uniform convergence $u_j \to u$, with u satisfying the same gradient estimates. From the first estimates in Lemma 3.9 we know that u is non-negative and that $u \to \infty$ in the asymptotic region (since the subsolution must). Since the regularised solutions are in fact strong IMCF solutions (with $U_j(x,\zeta) = (u_j(x) - \zeta)/\epsilon_j$) on $M \times \mathbb{R}$, they are also weak IMCF solutions and therefore the gradient estimate and Theorem 3.11 imply that their limit $U(x,\zeta) = u(x)$ is a weak IMCF solution on $M \times \mathbb{R}$. If U describes a smooth flow we are done - since the level sets of U are vertical cylinders, the principal curvature in the vertical direction is zero so the mean curvature is unchanged when we intersect with M. Since the velocity and normal vectors would also be unchanged, the smooth IMCF equation would be satisfied. For the general case, take any variational competitor v and let $V(x,\zeta) = v(x)\phi(\zeta)$ where

$$\phi(\zeta) = \begin{cases} 1 & \zeta \in [0,S] \\ \zeta + 1 & \zeta \in (-1,0) \\ S + 1 - \zeta & \zeta \in (S, S+1) \\ 0 & \zeta \notin (-1, S+1) =: I_S \end{cases}$$

is a Lipschitz cutoff function. Since U is a weak IMCF solution we have $J_U(U) \le J_U(V)$; i.e.

$$\int_{K \times I_S} |\nabla u| + u|\nabla u| \le \int_{K \times I_S} |\nabla V| + V|\nabla u| \le \int_{K \times I_S} \phi|\nabla v| + |\phi'| + v\phi|\nabla u|$$

for any K containing $\{u \ne v\}$. In the limit $S \to \infty$, the interval $[0,S]$ dominates the integral after we divide by S; i.e.

$$\frac{1}{S}\int_{K \times I_S} \phi|\nabla v| + |\phi'| + v\phi|\nabla u| \to \frac{1}{S}\int_{K \times I_S} |\nabla v| + v|\nabla u|;$$

so we have $\int_K |\nabla u| + u|\nabla u| \le \int_K |\nabla v| + v|\nabla u|$; i.e. u is a weak IMCF solution on M. □

Thus we have existence of the desired flow with $N_t = \partial\{u < t\}$.

3.4 Monotonicity

We have already made an argument for the monotonicity in the weak case - on surfaces with $\nabla u \ne 0$ everywhere we have smooth IMCF and thus monotonicity by Proposition 2.5, and at jumps $E_t \to E_t^+$ we have monotonicity by Corollary 3.7. If we could somehow determine that the jumps occur discretely in t, then we would have a watertight proof; but it is not immediately obvious how to do this. We instead will prove the monotonicity using the regularised solutions from the previous subsection.

Theorem 3.13. *For a weak IMCF solution, the level sets N_t satisfy*

$$m_H(N_s) \geq m_H(N_r) + \frac{1}{\sqrt{16\pi}} \int_r^s \sqrt{A(N_t)} \left(1 - \frac{1}{2}\chi(N_t) + \frac{1}{16\pi}\int_{N_t} 2|D\log H|^2 + (\lambda_1-\lambda_2)^2 + R\,dA\right) dt;$$

for all $s > r \geq 0$. In particular, the Geroch monotonicity $dm_H(\partial E_t)/dt \geq 0$ holds where $m_H(N_t)$ is differentiable and N_t are connected.

Proof. For any $\epsilon > 0$, let N_t^ϵ be the graph of $(u^\epsilon - t)/\epsilon$ in $M \times \mathbb{R}$ where u_ϵ is a solution to the regularised equation (3.9). Since N_t^ϵ is a *smooth* IMCF, we have by the computations in Proposition 2.5:

$$\frac{d}{dt}\int_{N_t^\epsilon} H^2 = \int_{N_t^\epsilon} \left(-2H\Delta\left(\frac{1}{H}\right) - 2\mathrm{Rc}(\nu,\nu) + H^2 - 2|A|^2\right) dA$$

where dA now denotes the 3-dimensional hyperarea. By the existence proof for (3.9), we have sequences $L_j \to \infty$, $\epsilon_j \to 0$ such that $N_t^j := N_t^{\epsilon_j} \to N_t \times \mathbb{R}$, and the mean curvature $H > 0$. We would like to take the limit of the integral equation above; but clearly the integrals will not be finite (since N_t are infinite vertical cylinders). Since what we are actually interested in is the 2-dimensional integrals after intersecting with M, we multiply by some cutoff function $\phi \in C_c^2(\mathbb{R})$ with $\phi \geq 0$, $\int \phi = 1$, $\mathrm{supp}\phi \subset [a,b]$ similarly to in the proof that the cylinder cross-sections solve the weak IMCF. Fix a $T > 0$ and move far enough along the sequence that $\sup u_\epsilon = L - 2 \geq T + b$ and $\epsilon \leq 1$ so that ∂N_t^ϵ (which is nonempty due to the regularised solutions existing only on a compact domain Ω_L) is disjoint from $\Omega = M \times [a,b]$. Using (2.3), we have

$$\frac{d}{dt}\int_{N_t^\epsilon} \phi H^2 = \int_{N_t^\epsilon} \phi\left(-2H\Delta\left(\frac{1}{H}\right) - 2\mathrm{Rc}(\nu,\nu) - 2|A|^2\right) dA + H^2\frac{\partial\phi}{\partial t}dA + \phi H^2 L_{\partial_t} dA.$$

Since $\partial_t = \nu/H$ and $L_{\partial_t}dA = dA$, this becomes

$$\frac{d}{dt}\int_{N_t^\epsilon} \phi H^2 = \int_{N_t^\epsilon} \left[\phi\left(-2H\Delta\left(\frac{1}{H}\right) - 2\mathrm{Rc}(\nu,\nu) - 2|A|^2 + H^2\right) + H\nabla_\nu\phi\right] dA.$$

Integrating by parts and then integrating this over a time interval $[r,s] \subset [0,T]$ we arrive at

$$\int_{N_s^\epsilon} \phi H^2 dA = \int_{N_r^\epsilon} \phi H^2 dA + \int_s^r \int_{N_t^\epsilon} \left[\phi\left(2\frac{|DH|^2}{H^2} + 2\mathrm{Rc}(\nu,\nu) + 2|A|^2 - H^2\right) + 2\frac{\langle D\phi, DH\rangle}{H} - H\nabla_\nu\phi\right] dA\,dt \tag{3.10}$$

where D is the covariant derivative on the hypersurface.

Huisken and Ilmanen derive estimates for all of the quantities in the above integrals, with resulting limits (after passing to some subsequence):

$$\begin{aligned}
\int_{N_t^j} \phi H^2 dA\, dt &\to \int_{N_t\times\mathbb{R}} \phi H^2 dA\, dt \\
\int_s^r \int_{N_t^j} \phi H^2 dA\, dt &\to \int_s^r \int_{N_t\times\mathbb{R}} \phi H^2 dA\, dt \\
\int_s^r \int_{N_t^j} |H\nabla_\nu \phi|\, dA\, dt &\to 0 \\
\int_s^r \int_{N_t^j} \frac{\langle D\phi, DH\rangle}{H} dA\, dt &\to 0 \\
\liminf_{j\to\infty} \int_r^s \int_{N_t^j} \phi \frac{|DH|^2}{H^2} dA\, dt &\geq \int_r^s \int_{N_t\times\mathbb{R}} \phi \frac{|DH|^2}{H^2} dA\, dt \\
\liminf_{j\to\infty} \int_r^s \int_{N_t^j} |A|^2\, dA\, dt &\geq \int_r^s \int_{N_t\times\mathbb{R}} |A|^2\, dA\, dt
\end{aligned}$$

(The second fundamental form of a C^1 set can be defined, and behaves as usual including the Gauss-Bonnet theorem for $\kappa = \det A$.) These are all intuitive remembering that $\nabla\phi$ is vertical while the geometry approaches vertical symmetry; see [1, Section 5] for the details. Thus we can take limits of (3.10), giving

$$\int_{N_s\times\mathbb{R}} \phi H^2 dA \leq \int_{N_r\times\mathbb{R}} \phi H^2 dA + \int_s^r \int_{N_t\times\mathbb{R}} \phi \left(2\frac{|DH|^2}{H^2} + 2\mathrm{Rc}\,(\nu,\nu) + 2\,|A|^2 - H^2 \right) dA\, dt.$$

Now that our integrals are over cylinders, each integral splits as

$$\int_{N_s\times\mathbb{R}} \phi\,(\zeta)\, f\,(x)\, dA\,(x)\, d\zeta = \int_{N_s} f dA \int_{\mathbb{R}} \phi = \int_{N_s} f dA$$

since the geometric quantities f have vertical symmetry and we chose $\int \phi = 1$. (Here dA is once again the 2-dimensional surface area measure.) Thus we have

$$\int_{N_s} H^2 dA \leq \int_{N_r} H^2 dA + \int_s^r \int_{N_t} \left(2\frac{|DH|^2}{H^2} + 2\mathrm{Rc}\,(\nu,\nu) + 2\,|A|^2 - H^2 \right) dA\, dt.$$

Similarly to the smooth case, we have

$$2\,|A|^2 + 2\mathrm{Rc}\,(\nu,\nu) - H^2 = \frac{1}{2}\,(\lambda_2 - \lambda_1)^2 + R - 2\kappa + \frac{1}{2} H^2$$

and thus applying the Gauss-Bonnet formula $\int_{N_t} \kappa = 2\pi\chi\,(N_t)$ (which is valid by approximation of N_t by C^2 surfaces [1]) and noting $|DH|/H = |D\log H|$ since $H > 0$, we have the desired result after some manipulation. □

3.5 Asymptotics

The asymptotic behaviour of $m_H\,(N_t)$ can be analysed using a *blowdown argument*. For a weak IMCF solution u and a $\lambda > 0$, the geometry is scaled as $\Omega^\lambda = \{\lambda x | x \in \Omega\}$, $g^\lambda\,(x) = \lambda^2\,(x/\lambda)$ and $u^\lambda = u\,(x/\lambda)$ where Ω is the asymptotically flat end of M (so it is covered by a single asymptotically flat coordinate chart with respect to which we perform the scalings).

Lemma 3.14. *There are some constants $c_\lambda \to \infty$ such that $u^\lambda\,(x) - c_\lambda \to (n-1)\log|x|$ loc-*

ally uniformly; i.e. modulo uniform relabellings of the level sets, the solution converges to the exponentially expanding spheres on $\mathbb{R}^n$.

Once this is established, we have convergence of the scaled-down level sets $N_t^{1/r(t)} \to \partial B_1(0)$ for $r(t)$ such that $A(N_t) = 4\pi r^2$. The following theorem is then proven by writing $N_t^{1/r(t)}$ as a graph over $\partial B_1(0)$, linearising the expression for the mean curvature of N_t and comparing to the (scaled-down) ADM mass integral. (See [1] Lemma 7.4 for details.)

Theorem 3.15. *If u solves the weak IMCF, then* $\lim_{t\to\infty} m_H(\partial E_t) \le m_{\mathrm{ADM}}$.

3.6 Rigidity

By Theorem 3.12, there is a solution $N_t = \partial E_t = \partial\{u < t\}$ of the weak IMCF starting at the horizon N_0, with $m_H(N_t)$ increasing by 3.13 and $\lim_{t\to\infty} m_H(N_t) \le m_{\mathrm{ADM}}$ by Theorem 3.15; so we have $m_{\mathrm{ADM}} \ge m_H(N_0) = \sqrt{\frac{A(N_0)}{16\pi}}$ since N_0 is minimal. Thus we have proven all of Theorem 1.12 except for the rigidity claim. The argument is very similar to the smooth case after some initial analysis.

Proposition 3.16. *If $m_{ADM} = \sqrt{A(N_0)/16\pi}$ then (M, g) is isometric to the Schwarzschild slice.*

Proof. By Theorems 3.13 and 3.15, we must have $m_H(N_t) = m_{\mathrm{ADM}}$ for all time. For equality to hold in Theorem 3.13, we must have

$$\int_{N_t} |D \log H|^2 = 0$$

for almost every t; thus H is constant on N_t for every t by Proposition D.8 (taking the limit $N_s \to N_t$, $s \nearrow t$). If there was a jump $N_t^+ \ne N_t$, then since $m_H(N_t^+) = m_H(N_t)$, the calculations in 3.7 imply that $H = 0$ on $N_t \backslash N_t^+$. But since H is constant on N_t, this implies N_t is a compact minimal surface (that does not touch N_0), contradicting the fact that N_0 is the outermost horizon. Thus $N_t = N_t^+$ and $H > 0$ for all t, so the flow can always be continued smoothly. The result now follows by Proposition 2.7. $\square$

4 Generalisations and other applications of IMCF

4.1 Multiple Horizons

It is physically reasonable to expect that if there are multiple black holes in the universe (i.e. ∂M is the disjoint union of more than one topological 2-sphere), then the inequality should still hold for $A(\partial M)$; this is the full Penrose inequality of Theorem 1.11. While the weak flow of Huisken and Ilmanen can be generalised [1, 6.1] to this case with the initial condition being one connected component N_0 of ∂M and the flow jumping over the other components at appropriate times, all it proves is $\sqrt{A(N_0)/16\pi} \leq m_{\mathrm{ADM}}$. If we try to use all of ∂M as our initial condition, then we do not have connectedness and therefore the monotonicity fails. We could instead start the flow with one connected component of the boundary and attempt to account for the areas of the other components by a more careful analysis of the jump times; but this is doomed to fail, as the following counterexample shows.

Example 4.1. Consider N_0 a tiny sphere in the Schwarzschild slice with centre a large distance from the origin. Then the Hawking mass of N_0 is tiny (since it is zero for Euclidean spheres), and the flow N_t will proceed smoothly for some finite time before jumping to encompass the Schwarzschild horizon. Since the spheres N_t are not the symmetric spheres of the Schwarzschild slice, the rigidity theorem implies that $m_H(N_t)$ is strictly increasing in the initial smooth flow. Thus we have $m_H(N_T) > 0$ for T the jump time, and $m_H\left(N_T^+\right) \leq m_{\mathrm{ADM}}$ by a similar argument ($m_H(N_t)$ must strictly increase while flowing out to infinity by rigidity). Therefore the mass m_{ADM} of the black hole is not just picked up in the jump that encompasses the singularity (as is the case when we start the flow on the horizon); it is somehow non-locally distributed, with an increase of only $m_H\left(N_T^+\right) - m_H(N_T) < m_{\mathrm{ADM}}$ at the jump time.

The generalised inequality $\sqrt{A(\partial M)/16\pi} \leq m_{\mathrm{ADM}}$ was eventually proven by Bray [12] by a different method, with the Positive Mass Theorem being a key ingredient in the proof. However, this does not obsolete the inverse mean curvature flow, as we will see in the following subsections.

4.2 Penrose Inequality with Charge

A natural extension of the Penrose inequality is to the case of a charged black hole. The Schwarzschild solution saturates the Penrose inequality, so to find a candidate bound for the horizon area we should look to the analogous solution for the Einstein-Maxwell equations - the Reissner–Nordström slice

$$g = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

where Q is the total charge and we use units where $4\pi\epsilon_0 = 1$. As in Example 2.4, we can compute the mean curvature by

$$H(S_r) = \frac{1}{|\partial_r|}\frac{d\ln A(S_r)}{dr} = \frac{2}{r}\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}.$$

Solving for $H(S_R) = 0$ gives us $R_\pm = m \pm \sqrt{m^2 - Q^2}$, so the slice has outermost horizon

$$R_+ = m + \sqrt{m^2 - Q^2}. \tag{4.1}$$

The Hawking masses of the coordinate spheres are

$$\begin{aligned} m_H(S_R) &= \frac{R}{2}\left(1-\frac{1}{16\pi}\int_{S_R} H^2 dA\right) \\ &= \frac{R}{2}\left(1-\frac{4\pi R^2}{16\pi}\frac{4}{R^2}\left(1-\frac{2m}{R}+\frac{Q^2}{R^2}\right)\right) \\ &= m-\frac{Q^2}{2R}. \end{aligned}$$

Since the perturbation to the Schwarzschild metric is at order r^{-2}, we still have $m_{\text{ADM}} = m$. Writing this in terms of the outermost horizon area $A_+ = 4\pi R_+^2$ and the total charge Q gives

$$m_{\text{ADM}}\,(\text{Reissner–Nordström}) = \sqrt{\frac{A_+}{16\pi}} + Q^2\sqrt{\frac{\pi}{A_+}}.$$

We see from this example that even in the Reissner-Nordström case, the Hawking mass must strictly increase as the test surface flows outwards from the horizon to the asymptotic region, unlike the uncharged case where the Hawking mass is constant while traversing Schwarzschild space. This reflects the scalar curvature term $2\left(|E|^2+|B|^2\right)$ contributed by the electromagnetic field. Thus we must not discard the scalar curvature term in our monotonicity calculation if we wish it to hold for the charged case.

Theorem 4.2. *If (M,g) is an asymptotically flat manifold of ADM mass m_{ADM} equipped with a vector field E and a positive real Q such that*

$$Q \leq \frac{1}{4\pi}\int_\Sigma \langle E, \nu_{S_r}\rangle\, dA$$

for any topological 2-sphere Σ containing the outermost horizon and $R \geq 2|E|^2$, then

$$m_{\text{ADM}}(M) \geq \sqrt{\frac{A_+}{16\pi}} + Q^2\sqrt{\frac{\pi}{A_+}}$$

where A_+ is the area of any connected component of the outermost horizon of M.

Note. The conditions on E mean physically that there is a total charge of at least Q inside the horizon, and the scalar curvature condition means that the local energy density is at least that contributed by the electromagnetic field, which is $R = 2\left(|E|^2+|B|^2\right)$ in the electrovacuum case.

Proof. Exactly as for the uncharged case, but with a more careful calculation of the evolution of $m_H(N_t)$. We estimate the scalar curvature term using the Cauchy-Shwarz and Hölder inequalities:

$$\begin{aligned} \int_{N_t} R\, dA &\geq 2\int_{N_t} |E|^2\, dA \\ &\geq 2\int_{N_t} \langle E, \nu_{N_t}\rangle^2\, dA \\ &\geq \frac{2\left(\int_{N_t}\langle E, \nu_{N_t}\rangle\, dA\right)^2}{A(N_t)} \\ &\geq \frac{2}{A(N_t)}(4\pi Q)^2 \end{aligned}$$

and thus using Theorem 3.13 with $r=0$, $s\to\infty$ (remembering $\chi(N_t)\le 2$) we find

$$\begin{aligned}
m_{\mathrm{ADM}} &\geq m_H(N_0)+\frac{1}{\sqrt{16\pi}}\int_0^\infty \sqrt{A(N_t)}\left(1-\frac{1}{2}\chi(N_t)+\frac{1}{16\pi}\int_{N_t}2\left|D\log H\right|^2+(\lambda_1-\lambda_2)^2+R\,dA\right)dt\\
&\geq \sqrt{\frac{A_+}{16\pi}}+\frac{1}{(16\pi)^{3/2}}\int_0^\infty\sqrt{A(N_t)}\int_{N_t}R\,dA\,dt\\
&\geq \sqrt{\frac{A_+}{16\pi}}+\frac{1}{(16\pi)^{3/2}}\int_0^\infty\frac{2}{\sqrt{A(N_t)}}(4\pi Q)^2\,dt\\
&\geq \sqrt{\frac{A_+}{16\pi}}+\frac{Q^2\sqrt{\pi}}{2\sqrt{\pi A_+}}\int_0^\infty e^{-t/2}dt\\
&= \sqrt{\frac{A_+}{16\pi}}+Q^2\sqrt{\frac{\pi}{A_+}}.
\end{aligned}$$

□

Bray's proof admits no such simple generalisation. In fact, the Penrose inequality with charge and disconnected horizon is false - see [15] for a counterexample. Thus the IMCF proof of the Penrose Inequality still has merits over Bray's approach.

4.3 The Yamabe Invariant of $\mathbb{RP}^3$

Another application of the IMCF arises in the study of the Yamabe Invariant, a (smooth) topological invariant arising from the study of conformal transformations in differential geometry.

Throughout this section, let M be a compact smooth n-manifold, $n\geq 2$.

Definition 4.3. For a Riemannian metric g on M, the *Einstein-Hilbert energy* $E(g)$ is

$$E(g)=\frac{\int_M R\,dV}{\left(\int_M dV\right)^{(n-2)/n}}$$

where R, dV are the scalar curvature and volume forms induced by the metric g.

Definition 4.4. The *smooth Yamabe invariant* or *σ-invariant* of a smooth manifold M is

$$\sigma(M)=\sup\{Y(g)\,|g\text{ is a metric on }M\}$$

where

$$Y(M,g)=Y(g)=\inf_{[g]}E$$

is the *conformal Yamabe invariant* of the conformal class $[g]=\{$metrics conformal to $g\}$.

An important question is whether or not the infimum $Y(g)$ is actually attained. A necessary condition for $E(g_0)=Y(g_0)$ is that g satisfies the Euler-Lagrange equation for the functional E (i.e. g_0 is a local minimum of E). By definition, $[g]$ is the set of metrics $\tilde{g}$ such that $\tilde{g}=u^{2/(n-2)}g$ for some smooth function u on M. The scalar curvature of $\tilde{g}$ is

$$\tilde{R}=u^{-(n+2)/(n-2)}L_0u \tag{4.2}$$

where

$$L_0=R_g-4\frac{n-1}{n-2}\Delta_g$$

is the *conformal Laplacian.* In terms of u and the geometry of g, the energy of $\tilde{g}$ is therefore (noting $dV_{\tilde{g}} = \sqrt{\frac{\det \tilde{g}}{\det g}} dV = u^{2n/(n-2)}$ and integrating by parts)

$$E(\tilde{g}) = E_g(u) = \frac{\int u L_0 u \, dV}{\left(\int u^{2n/(n-2)} dV\right)^{(n-2)/n}} = \frac{\int \left(uRu - 4\frac{n-1}{n-2} u \Delta u\right) dV}{\left(\int u^{2n/(n-2)} dV\right)^{(n-2)/n}} = \frac{\int \left(Ru^2 + 4\frac{n-1}{n-2} |\nabla u|^2\right) dV}{\left(\int u^{2n/(n-2)} dV\right)^{(n-2)/n}}. \tag{4.3}$$

Thus our question is whether or not E_g has a minimum over $C^\infty(M)$. The Euler-Lagrange equation (a necessary condition for a local minimum) of E_g is

$$L_0 u = \lambda u^{(n+2)/(n-2)}$$

where $\lambda = E_g(u) / \|u\|_{L^p}^{p-2}$; so if E_g is minimised at $g_0 = u_0^{2/(n-2)} g$ then (by (4.2)) g_0 has constant scalar curvature. The existence of such a u_0 is a known as the Yamabe Problem, conjectured and thought to be proven by Yamabe [16]; however, his proof had a major flaw. The work of Trudinger, Aubin and Schoen eventually culminated in a correct proof [17]; so we know that in each conformal class $[g]$, $Y(g) = E(g_0)$ for some g_0 with constant scalar curvature.

The supremum $\sigma(M)$ is defined due to the following lemma. The proof involves constructing a metric conformal to g which looks like g_0 over most of S^n in a neighbourhood of a point, and is very small everywhere else in M.

Lemma 4.5. *$Y(M, g) \leq E(g_0)$ where g_0 is the standard round metric on S^n.[18]*

A good starting point for computing the invariant is the following theorem.

Theorem 4.6. *(Obata [18, 19]). If g is Einstein ($Rc = kg$ for some constant k) then $Y(g) = E(g)$.*

In the two-dimensional case, the scalar curvature is twice the Gaussian curvature so we have

$$E(g) = \int_M 2K \, dA = 4\pi\chi(M)$$

which is independent of g and therefore $\sigma(M) = 4\pi\chi(M)$; i.e. the Yamabe invariant reduces to the well-studied Euler characteristic.

Much more interesting is the three-dimensional case, which is where we will find an application of the IMCF. The starting point is the example S^3, where (by Lemma 4.5) the standard round metric (from the embedding $S^3 \hookrightarrow \mathbb{R}^4$ with radius 1) achieves the maximum. Since it has constant scalar curvature 6 and volume $2\pi^2$, we see that

$$\sigma_1 := \sigma\left(S^3\right) = \frac{\int_{S^3} 6 dV}{\left(\int_{S^3} dV\right)^{1/3}} = 6\left(2\pi^2\right)^{2/3}.$$

For general 3-manifolds we have $\sigma(M) \leq \sigma_1$; but finding exact values is difficult. The Yamabe invariant $\sigma\left(\mathbb{RP}^3\right)$ was finally computed by Bray and Neves in 2004 using the weak IMCF of Huisken and Ilmanen along with the relationship between the conformal Yamabe invariant $Y(g)$ and the optimal constant for the Gagliardo-Nirenberg-Sobolev inequality.

Theorem 4.7. $\sigma\left(\mathbb{RP}^3\right) = \sigma_2 := 6\pi^{4/3}$.

Proof. We give a brief sketch; see [18] for the full details (including a more general classification result).

We have $\sigma\left(\mathbb{RP}^3\right) \geq \sigma_2$ by the theorem of Obata [19]. Thus what needs to be shown is that $Y(g) \leq \sigma_2$ for some g in each conformal class of metrics on $\mathbb{RP}^3$.

Consider a conformal class $[g]$ with $Y(g) > 0$; then the minimum is attained $(E(g_0) = Y(g_0))$ at some metric $g_0 \in [g]$ of positive constant scalar curvature R. Fix a point $p \in M$; then there is a some positive scaling of the fundamental solution G_p of the conformal Laplacian on $(M \backslash p, g_0)$ such that the metric $g_{AF} = G_p^4 g_0$ is asymptotically flat on $M \backslash \{p\}$ (with the asymptotic regime being $x \to p$). The new metric g_{AF} is in the conformal class $[g]$, and since $L_0 G_p = 0$ on $M \backslash \{p\}$, g_{AF} has zero scalar curvature by (4.2). Therefore by (4.3) we have

$$Y(g_0) = Y(g_{AF}) = \inf\left\{ \frac{\int_M 8 |\nabla u|^2 \, dV}{\left(\int_M u^6 \, dV\right)^{1/3}} \middle| u \in C^\infty(M \backslash \{p\}) \right\}$$

where the derivatives and norms are with respect to g_{AF}. If we change the conformal factors to be H^1 with compact support, this ratio (without the factor of 8) is in fact the optimal constant $S(g_{AF})$ in the inequality on $(M \backslash \{p\}, g_{AF})$; and it can be shown by approximation that the infimum is the same for both classes of functions. Thus we have reduced the problem to showing that $S(g_{AF}) \le \sigma_2/8$.

In the case where our original conformal class is $[g] = [g_0] = [g_R]$ with g_R being the round metric on $\mathbb{RP}^3$ (the projection of the round metric on S^3 down the double covering $S^3 \to \mathbb{RP}^3$, which exists because the antipodal map is an isometry), the symmetries of g_R imply that g_{AF} has spherical symmetry (i.e. its isometry group has a subgroup isometric to $SO(3)$). Consider the lift of g_{AF} to S^3. Since it is spherically symmetric, has zero scalar curvature and is geodesically complete, it must be isometric to some scaling of the Schwarzschild slice $\left(M_S = \mathbb{R}^3 \backslash B_1(0), g_S\right)$; and therefore $\left(\mathbb{RP}^3, g_{\mathrm{AF}}\right)$ is isometric to the exterior region of M_S (with antipodal points on the horizon $\partial B_1(0)$ identified) [2, 18]. In this case we also have $S(g_{AF}) \le \sigma_2/8$ by Theorem 4.6.

To prove this inequality for other initial metrics, we will compare back to g_R and apply the IMCF. First, define the function U_0 on M_S so that $g_R = U_0^4 g_S$. We know that g_R minimises E over the conformal class and therefore that U_0 attains the optimal Sobolev constant of $\sigma_2/8$; and U_0 also has spherical symmetry. Let N_t be the IMCF starting at the horizon in M_S; then N_t is just a flow of concentric spheres, so by the spherical symmetry of U_0 we can define $f(t) = U_0(N_t)$.

Now consider the general manifold $\left(\mathbb{RP}^3 \backslash \{p\}, g_{AF}\right)$. One can show that there is an outermost compact minimal surface; so letting this be the initial condition we find (by Theorem 3.12) a weak IMCF solution u. We will use

$$U := \begin{cases} f \circ u & \text{on } \{u > 0\} \\ f(0) & \text{on } \{u \le 0\} \end{cases}$$

as a test function for the Sobolev inequality and compute the resulting ratio using properties of the IMCF. The numerator of the Sobolev ratio is (by the co-area formula for the slices $N_t = \partial \{u < t\}$)

$$\int_M |\nabla U|^2 \, dV = \int_0^\infty f'(t)^2 \int_{N_t} H \, dA \, dt.$$

The denominator can estimated using the co-area formula and the dominated convergence theorem by

$$\int_M |\nabla U|^6 \, dV \ge \int_0^\infty f(t)^6 A(N_t)^2 \left(\int_{N_t} H \, dA\right)^{-1} dt. \tag{4.4}$$

By the monotonicity of the Hawking mass and the exponential growth of surface area we have

$$\int_{N_t} H^2 \, dA \le 16\pi \left(1 - e^{-t/2}\right)$$

and thus by Hölder's inequality

$$\int_{N_t} H\,dA \leq \sqrt{16\pi A(N_0)\left(e^t - e^{t/2}\right)}.$$

If we substitute this into (4.4) we arrive at an estimate for the Sobolev constant:

$$S(g_{AF}) \leq \frac{\int |\nabla U|^2\,dV}{\left(\int U^6 dV\right)^{1/3}} \leq \frac{(16\pi)^{2/3}\int_0^\infty f'(t)\sqrt{e^t - e^{t/2}}dt}{\left(\int_0^\infty f(t)^6\,e^{2t}\left(e^t - e^{t/2}\right)^{-1/2}dt\right)^{1/3}} =: C.$$

Note now that the right hand side makes no reference to the geometry of g_{AF} - it is entirely defined in terms of the function f, which we defined from the model case when $g_{AF} \simeq g_S$. In the model case, the Hawking mass is constant and all the inequalities above are in fact equalities; so we find $C = S(g_S) = \sigma_2/8$. (Alternatively we find $f(t) = \left(2e^t - e^{t/2}\right)^{-1/2}$ from the expanding sphere solution of the IMCF on M_S and compute the integrals.) We therefore have in general

$$Y(g_{AF}) = 8S(g_{AF}) \leq \sigma_2.$$

□

A Asymptotics

A central concept throughout the paper is that of asymptotic flatness, and many of the calculations therefore involve taking limits $r \to \infty$. To make these calculations clean, we use the usual big-O notation for asymptotics. This appendix clearly defines this notation and the way we use it in the multivariable setting.

Assume we have some distinguished coordinate system x^1, x^2, x^3 on $M = \mathbb{R}^3 \backslash K$ where K is compact. Then

Definition A.1. The spherical coordinate system associated with $\left\{x^1, x^2, x^3\right\}$ is the one given by $\{r, \theta, \varphi\}$ satisfying

$$\begin{aligned} x^1 &= r \cos\varphi \sin\theta \\ x^2 &= r \sin\varphi \sin\theta \\ x^3 &= r \cos\theta; \end{aligned}$$

i.e. the usual spherical polar coordinates if $\left\{x^1, x^2, x^3\right\}$ are Cartesian coordinates on Euclidean space.

Definition A.2. A function $f : M \to \mathbb{R}$ is $O\left(r^k\right)$ as $r \to \infty$ if there exists an $R > 0$ and a $C > 0$ such that

$$r > R \implies \left|f\left(r, \theta, \varphi\right)\right| < Cr^k.$$

We will write this as $f \in O\left(r^k\right)$ or $f = O\left(r^k\right)$. It is often convenient to use the notation $f = g + O\left(r^k\right)$ to mean $f - g \in O\left(r^k\right)$.

In particular, the restricted functions $f\left(r, \cdot, \cdot\right) : \partial B_r \to \mathbb{R}$ are bounded by Cr^k for $r > R$; we will often use this to estimate integrals $\int_{\partial B_r} f \, dA$ as $r \to \infty$. An important consequence is that in asymptotically flat coordinates and when the sphere is parametrised with angular coordinates θ, φ we have $\iint f \sqrt{g} \, d\theta d\varphi$ with $\sqrt{g} \in O\left(r^2\right)$; so for $\alpha < -2$ the condition $f \in O\left(r^\alpha\right)$ guarantees the integral vanishes as $r \to \infty$.

For the purpose of computation, all the usual power-series manipulations are valid as long as we understand the coefficients as bounded functions of θ, φ.

B First Variation of Area

This equation is central to the geometric analysis of the IMCF.

Proposition B.1. *Let M be a Riemannian n-manifold. If $N_t = \Phi_t^X N_0 \subset M$ is a smooth family of hypersurfaces satisfying $X = v\nu$ (i.e. flowing in the normal direction with speed v) then*

$$L_X dA = vH\, dA$$

where dA is the area form induced by g on N_t, ν is the unit normal to N_t and H is the mean curvature scalar of N_t.

Proof. The flow gives us natural coordinates $\{x^1, \ldots, x^{n-1}, x^n = t\}$ so that $N_\tau = \{\tau = t\}$. Then we have $X = \partial_t$ and

$$dA = \sqrt{\det g_N} dx \wedge dy$$

and therefore by Cartan's magic formula

$$\begin{aligned} L_X dA &= (di_{\partial_t} + i_{\partial_t} d)\left(\sqrt{\det g_N} dx \wedge dy\right) \\ &= i_{\partial_t}\left(\frac{\partial\sqrt{\det g_N}}{\partial t} dt \wedge dx \wedge dy\right) \\ &= \frac{1}{2\sqrt{\det g_N}} \frac{\partial \det g_N}{\partial t} dx \wedge dy \\ &= \frac{1}{2\det g_N} \frac{\partial \det g_N}{\partial t} dA. \end{aligned}$$

By the formula for the derivative of the determinant, this is

$$\begin{aligned} L_X dA &= \frac{1}{2}\mathrm{tr}\left(g_N^{-1}\frac{\partial g_N}{\partial t}\right) dA. \\ &= \frac{1}{2}\sum_{i,j=1}^{n-1} g^{ji} g_{ij,t} dA \end{aligned}$$

We now compute the derivatives of the tangential metric components: for $i, j \leq n-1$,

$$\begin{aligned} g_{ij,t} &= \nabla_t \langle \partial_i, \partial_j \rangle \\ &= v\nabla_\nu \langle \partial_i, \partial_j \rangle \\ &= v\left(\langle \nabla_\nu \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_\nu \partial_j \rangle\right) \\ &= v\left(\langle [\nu, \partial_i] + \nabla_i \nu, \partial_j \rangle + \langle \partial_i, [\nu, \partial_j] + \nabla_j \nu \rangle\right). \end{aligned}$$

Since $[\nu, \partial_i] = \left[\frac{1}{v}\partial_t, \partial_i\right] = -\left(\partial_i v\right)\partial_t$ and $\langle \partial_t, \partial_i \rangle = 0$, the two Lie bracket terms disappear; and recognising the second fundamental form in the leftover terms we see $g_{ij,t} = 2vA_{ij}$. Therefore

$$L_X dA = \frac{1}{2} g^{ij} 2vA_{ij} dA = Hv\, dA.$$

□

C Computation

Prop 2.6

The following Mathematica code computes the Hawking mass of coordinate spheres in an asymptotically (and conformally to 1st order in r^{-1}) flat manifold.

```
(* Define Coordinate Transformation + Compute Jacobian *)
X = { r Cos[phi] Sin[theta],
   r Sin[phi] Sin[theta], r Cos[theta] };
J = Transpose[D[X, #] & /@ {r, phi, theta}];
(* Define metric in asymptotically Schwarzschild coordinates *)
h = Table[O[r, Infinity]^2 , {i, 1, 3}, {j, 1, 3}];
g = (1 + m/(2 r))^4 IdentityMatrix[3] + h;
(* Compute metric in new coordinates *)
gS = Transpose[J].g.J // FullSimplify;
(* Compute induced metric on coordinate sphere *)
gS2 = Take[gS, {2, 3}, {2, 3}];
(* Compute area of coordinate sphere *)
A = Integrate[
   Sqrt[Det[gS2]], {theta, 0, Pi}, {phi, 0, 2 Pi}];
(* Construct unit normal n to coordinate sphere *)
proj[v_, u_] := u (gS.u.v)/(gS.u.u);
pr = {1, 0, 0}; pphi = {0, 1, 0}; ptheta = {0, 0, 1};
uphi = pphi;
utheta = ptheta - proj[ptheta, uphi];
ur = pr - proj[pr, uphi] - proj[pr, utheta];
n = ur/Sqrt[gS.ur.ur];
(* Derivatives of metric *)
dg[i_, j_, k_] :=
  Sum[Inverse[J][[l, k]] D[
     g[[i, j]], {r, phi, theta}[[l]]], {l, 1, 3}];

(* Christoffel Symbols *)
G = Table[Sum[
      1/2 Inverse[g][[i, l]] (-dg[j, k, l] + dg[k, l, j] +
         dg[l, j, k]), {l, 1, 3}], {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]; //
   FullSimplify;
(* Compute mean curvature from covariant derivative *)
H = Sum[D[(J.n)[[i]], {r, phi, theta}[[j]]] Inverse[
        J][[j, i]], {i, 1, 3}, {j, 1, 3}] +
    Tr[G.(J.n)] // FullSimplify;
(* Compute Hawking mass of coordinate sphere *)
mH = Sqrt[A/(16 Pi)] (1 - (1/(16 Pi)) Integrate[
        H^2  Sqrt[Det[gS2]], {theta, 0, Pi}, {phi, 0,
         2 Pi}]) // FullSimplify
```

The result is $m_H(r) = m + O(1/r)$, allowing us to conclude the result of Proposition 2.6.

Example 3.8

The following Mathematica code computes the jump radius r_{jump} and produces the figures seen in Example 3.8.

```
f[a_, x_] := Cosh[a x]/a;
g[r_, x_] := Sqrt[r^2 - (x - 1)^2];
sol[r_] := FindRoot[{f[a, x] - g[r, x],
    D[f[a, x], x] - D[g[r, x], x]}, {x, 0.5}, {a, 1}];
catarea[a_, x0_] :=
  2 \[Pi] NIntegrate[f[a, x] Sqrt[1 + D[f[a, x], x]^2], {x, 0, x0}];
caparea[r_, x0_] :=
  2 \[Pi] NIntegrate[
    g[r, x] Sqrt[1 + D[g[r, x], x]^2], {x, 1 - r, x0}];
areagain[r_] :=
  catarea[a /. sol[r], x /. sol[r]] - caparea[r, x /. sol[r]];
solfn[r_, a_, x0_] := If[Abs[#] < x0, f[a, #], g[r, #]] &;
splot[r_, a_, x0_] :=
  ParametricPlot3D [{x, Cos[y] h[Abs[x]], Sin[y] h[Abs[x]]} /.
    h -> solfn[r, a, x0], {x, -2, 2}, {y, 0, 2 \[Pi]}];
splot2[r_, a_, x0_] :=
  Plot[solfn[r, a, x0][Abs[x]], {x, -2, 2},
   PlotRange -> {{-2, 2}, {0, 1}}];
solplot[r_] := splot[r, a /. sol[r], x /. sol[r]];
solplot2[r_] := splot2[r, a /. sol[r], x /. sol[r]];
rjump = r /. FindRoot[areagain[N[r]], {r, 0.86}, Evaluated -> False]
aminfn[r_] :=
  If[r >= rjump  , solfn[r, a /. sol[r], x /. sol[r]], (g[r, #] &)];
aminplot2[r_] :=
  Plot[aminfn[r][Abs[x]], {x, -2, 2}, PlotRange -> {{-2, 2}, {0, 1}}];

Plot[areagain[r], {r, 0.85, 1}, AxesLabel -> {"r", "\[Delta]A"}]

rvals = Append[Table[Exp[t], {t, -0.55, -0.15, 0.05}], rjump];
Show[Table[aminplot2[r], {r, rvals}]]

solplot[rjump]
```

D Geometric Measure Theory

While we omit many details, here are some basic definitions and results from measure theory that are integral to the variational formulation of the IMCF. See e.g. [13] for a detailed treatment.

Definition D.1. The *d-dimensional Hausdorff outer measure* of a set $E \subset M$ (M a metric space) is

$$\mathcal{H}^d(E) := \lim_{\delta \to 0} \inf \left\{ \alpha_d \sum_{n \in \mathbb{N}} r_n^d \middle| \{B(p_n, r_n < \delta)\}_{n \in \mathbb{N}} \text{ covers } E \right\}$$

where α_d is the usual measure (area, volume, etc.) of a d-dimensional sphere. (**N.B.** Some authors instead use $\alpha_d = 1$ or $\alpha_d = 2^d$; our definition corresponds to the classical notions of area and volume.) A set E is *$\mathcal{H}^d$-measurable* (and we call $\mathcal{H}^d(E)$ the *Hausdorff measure* of E) if for every set F we have

$$\mathcal{H}^d(F) = \mathcal{H}^d(E \cap F) + \mathcal{H}^d(E^c \cap F).$$

When restricted to the σ-algebra of measurable sets, the Hausdorff measure is countably additive. The n-dimensional Hausdorff measure on an n-manifold agrees with the Lebesgue measure. In the setting of 3-manifolds, we refer to $V = \mathcal{H}^3$ as volume and $A = \mathcal{H}^2$ as surface area. Integrals of functions are taken with respect to dV unless otherwise noted.

Definition D.2. A function $u : \Omega \to \mathbb{R}$ has *bounded variation in* Ω ($u \in BV(\Omega)$) if its *total variation*

$$\|\nabla u\| = \int_\Omega |\nabla u| := \sup \left\{ \int_\Omega u \operatorname{div} X \middle| X \in C_c^1(\Omega, T\Omega), \, \||X|\|_\infty \leq 1 \right\}$$

is finite. The ∇u appearing here is the vector measure corresponding to the distributional derivative of u, and $|\nabla u|$ is defined by the above supremum. A function $u : \Omega \to \mathbb{R}$ has *locally bounded variation* ($u \in BV_{\text{loc}}(\Omega)$) if $u \in BV(U)$ for each precompact open $U \subset \Omega$.

-

Definition D.3. The *perimeter* (or often *surface area*) of a set E in Ω is the total variation of its characteristic function in Ω:

$$P(E, \Omega) = \int_\Omega |\nabla \chi_E|.$$

E is said to have *locally finite perimeter* if $\chi_E \in BV_{\text{loc}}$. The outwards unit normal to ∂E is the vector field ν such that $\nabla \chi_E = -|\nabla \chi_E| \nu$; i.e. the ν making the divergence theorem

$$\int_\Omega X \cdot \nu \, |\nabla \chi_E| = \int_E \operatorname{div} X$$

hold.

Definition D.4. The *reduced boundary* $\partial^* E$ of a set of locally finite perimeter E is the set of points $x \in \partial E$ such that

$$\left| \lim_{\epsilon \to 0} \frac{\int_{B_\epsilon(x)} \nabla \chi_E}{\int_{B_\epsilon(x)} |\nabla \chi_E|} \right| = 1$$

The restriction of the Hausdorff measure $\mathcal{H}^{n-1}$ to $\partial^* E$ is exactly $|\nabla \chi_E|$; so $\mathcal{H}^{n-1}(\partial^* E) = P(E, \Omega)$ for $E \subset\subset \Omega$.

Definition D.5. If N is a C^1 hypersurface of M, then we say $H \in L^1_{\text{loc}}(N)$ is the *weak mean curvature* of N if

$$\frac{d}{dt}|_{t=0} A\left(\Phi_t^X(N) \cap W\right) = \int_{N \cap W} H \langle \nu, X \rangle \, dA$$

for every $X \in C_c^\infty(M, TM)$ and every precompact open $W \supset \operatorname{supp} X$.

Now that we have these definitions, we briefly review some useful results.

Theorem D.6. ***Co-area Formula.*** *For any $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and $v \in L^1(\Omega)$ we have*

$$\int_\Omega v\,|\nabla u|\,d\mathcal{H}^n = \int_{\mathbb{R}}\int_{u^{-1}(t)} v\,d\mathcal{H}^{n-1}\,dt.$$

Note that the integral is well-defined because locally Lipschitz functions are almost everywhere differentiable.

Proposition D.7. ***Lower Semi-Continuity of Surface Area****. If E_n, E are sets of finite perimeter with $\mathcal{H}^n(E_n \Delta E) \to 0$ or equivalently $\chi_{E_n} \xrightarrow{L^1} \chi_E$ as $n \to \infty$; then we have*

$$\mathcal{H}^{n-1}(\partial^* E) \leq \liminf_{n\to\infty} \mathcal{H}^{n-1}(\partial^* E_n).$$

Proposition D.8. ***(Semi-)Continuity properties of Weak Mean Curvature.*** *Under local C^1 convergence of C^1 hypersurfaces with $|H|$ uniformly bounded, $\int H\langle \nu, X\rangle$ is continuous for any X, $\operatorname{ess\,sup}|H|$ is lower semicontinuous and $\int \phi H^2$ is lower semicontinuous for smooth ϕ.*

E Notation

$X \subset\subset Y$	X is precompact and $\bar{X} \subset \mathrm{Int}(Y)$
$\langle \cdot, \cdot \rangle$	$g(\cdot,\cdot)$
$\lvert X \rvert$	$\sqrt{\lvert g(X,X) \rvert}$
$X^\flat$	1-form dual to vector field X with respect to g
$\theta^\sharp$	vector field dual to 1-form θ with respect to g
∇	(possibly distributional) covariant derivative
Δ	the covariant Laplacian
∂	boundary or reduced boundary
∂^*	reduced boundary
χ_E	indicator function of the set E
ν	unit normal vector to a hypersurface
A	second fundamental form of a hypersurface
A,dA	$\mathcal{H}^2$, $d\mathcal{H}^2$ when working in 3 dimensions
$\mathcal{A}(\Omega)$	a function space over Ω; i.e. vector subspace of $\mathbb{R}^\Omega$ (typically with some norm)
$\mathcal{A}_{\mathrm{loc}}(\Omega)$	functions $\Omega \to \mathbb{R}$ that have restrictions in $\mathcal{A}(U)$ for every $U \subset\subset \Omega$
$BV(\Omega)$	functions of bounded variation; i.e. with distributional derivatives in $\mathcal{D}'$
$C^k(\Omega)$	k-times continuously differentiable functions $\Omega \to \mathbb{R}$
$C^{k,\alpha}(\Omega)$	functions $\Omega \to \mathbb{R}$ with derivatives up to order k being Hölder continuous with exponent α
$C^\infty(\Omega)$	smooth functions on Ω
$C^\infty_c(\Omega)$ or $\mathcal{D}(\Omega)$	smooth functions with supp $\subset\subset \Omega$
$\mathcal{D}'(\Omega)$	distributions on Ω; real-valued Radon measures on Ω; the continuous dual of $\mathcal{D}(\Omega)$
D	covariant derivative on a submanifold
$i_X\omega$	interior product of form ω with vector field X; i.e. partial application $\omega(X,\cdot,\cdots)$
Φ^X_t	the flow of a vector field X with parameter t; i.e. the solution to $\frac{d}{dt}\Phi^X_t(p) = X\left(\Phi^X_t p\right)$
g	Riemannian metric
G	Einstein tensor ($\mathrm{Rc} - \frac{1}{2}Rg$)
$H^k(\Omega)$	$W^{k,2}(\Omega)$
$\mathcal{H}^n$	n-dimensional Hausdorff measure
k	second fundamental form of Lorentzian embedding $M \hookrightarrow L$
$L^p(\Omega)$	functions on Ω with $\lvert f \rvert^p$ integrable
λ_1, λ_2	principal curvatures of N; i.e. eigenvalues of A
L	Lorentzian 4-manifold
L_X	Lie derivative in the direction X
M	Riemannian 3-manifold
μ	4-dimensional hypervolume measure
N	2-dimensional hypersurface of M
$P(E,\Omega)$	perimeter of E in Ω ($\mathcal{H}^{n-1}(\partial^* E)$ when $E \subset\subset \Omega \subset \mathbb{R}^n$)
R	scalar curvature of M
Rc	Ricci curvature of M
Rm, R_{ijkl}	Riemannian curvature of M
T	stress-energy tensor
V,dV	$\mathcal{H}^3, d\mathcal{H}^3$ when working in 3 dimensions
$W^{k,p}(\Omega)$	(Sobolev space) L^p functions with distributional derivatives up to order k in L^p
$A \backslash B$	set difference $\{x \in A \mid x \notin B\}$
$Z_{(\mu\nu)}$	symmetrisation in indices μ,ν: $\frac{1}{2}(Z_{\mu\nu} + Z_{\nu\mu})$ (Z any tensor)
$Z_{[\mu\nu]}$	antisymmetrisation in indices μ,ν: $\frac{1}{2}(Z_{\mu\nu} - Z_{\nu\mu})$

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